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DETERMINATION OF LYAPUNOV EXPONENTS AND STUDY OF TIME-SERIES GRAPHS ON A NONLINEAR CHAOTIC MODEL

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ABSTRACT

In this paper, we study the chaotic model:

$$\rho(x) = ax^2 - bx$$

where $x \in [0,4]$, $a = -1$ and $b \in [-1, -4]$ is a tunable parameter and adopt the two techniques (i) Lyapunov Exponents and (ii) Time-series Analysis, in order to confirm the periodic orbits of periods 2^0, 2^1, 2^2… , as the parameter varies in a suitable region and the existence of the chaotic region. Finally some enlightening results have been achieved.

KEY WORDS: Lyaponuv exponents/ Periodic orbits / Time-series analysis/ Chaotic region

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INTRODUCTION

It has long been realized that the responses of many nonlinear dynamical systems do not follow simple, regular, and predictable trajectories, but swirl around in a random-like and seemingly irregular, yet well-defined, fashion. As long as the process involved is non- linear, even a simple strictly deterministic model may develop such complex behavior. This behavior is understood and accepted as chaos [5].

A chaotic system is one in which long-term prediction of the system's state is impossible because the omnipresent uncertainty in determining its initial state grows exponentially fast in time. The rapid loss of predictive power is due to the property that orbits (trajectories) that arise from nearby initial conditions diverge exponentially fast on the average.

Rates of orbital divergence or convergence, called Lyapunov exponents, are clearly a fundamental importance in studying chaos. Positive Lyapunov exponents indicate orbital divergence and chaos, and set the time scale on which state prediction is impossible.

Negative Lyapunov exponents set the time scale on which transients or perturbations of the system's state will decay [1, 2].

The key theoretical tool used for quantifying chaotic behavior is the notion of a time series of data for the system [7]. Orbit complexity is one of the interesting properties of chaotic systems. Orbit complexity means that chaotic systems contain an infinite number of unstable periodic orbits, which coexist with the strange attractor and play an important role in the system dynamics [6].

However, in many practical situations one does not have access to system equations and must deal directly with experimental data in the form of a time series. We now highlight some useful concepts which are absolutely useful for our purpose.

Discrete Dynamical Systems

Any $C^k(k \geq 1)$ map $E: U \to \Re^n$ on the open set $U \subset R^n$ defines an *n*-dimensional discrete-time (autonomous) smooth dynamical system by the state equation

$$\bar{x}_{t+1} = E(\bar{x}_t), t = 1,2,3,..... \quad (1.1)$$

where $\bar{x}_t \in \Re^n$ is the state of the system at time t and E maps $\bar{x}_t$ to the next state $\bar{x}_{t+1}$. Starting with an initial data $\bar{x}_0$, repeated applications (iterates) of E generate a discrete set of points (the orbits) $\{E^t(\bar{x}_0): t = 0,1,2,3,.....\}$, where $E^t(\bar{x}) = \underbrace{E \circ E \circ ... \circ E(\bar{x})}_{t \text{ times}}$ [9].

Definition: A point $\bar{x}^* \in \Re^n$ is called a fixed point of E if $E^m(\bar{x}^*) = \bar{x}^*$, for all $m \in$ ☪.

Definition: A point $\bar{x}^* \in \Re^n$ is called a periodic point of E if $E^q(\bar{x}^*) = \bar{x}^*$, for some integer $q \geq 1$.

Definition: The closed set $\boldsymbol{A} \in \Re^n$ is called the attractor of the system (1.1) if (i) there exists an open set $\boldsymbol{A}_0 \supset \boldsymbol{A}$ such that all trajectories $\bar{x}_t$ of system beginning in $\boldsymbol{A}_0$ are definite for all $t \geq 0$ and tend to $\boldsymbol{A}$ for $t \to \infty$, that is, $\text{dist}(\bar{x}_t, \boldsymbol{A}) \to 0$ for $t \to \infty$, if $\bar{x}_0 \in \boldsymbol{A}_0$, where

$$\text{dist}(\bar{x}, \boldsymbol{A}) = \inf_{\bar{y} \in \boldsymbol{A}} \|\bar{x} - \bar{y}\|$$

is the distance from the point $\bar{x}$ to the set $\boldsymbol{A}$, and (ii) no eigensubset of $\boldsymbol{A}$ has this property.

Definition: A system is called chaotic if it has at least one chaotic attractor.

Diffeomorphism: Let *A* and *B* are open subsets of $\Re^n$. A map $E: A \to B$ is said to be a diffeomorphism if it is a bijection and both E and E^{-1} are differentiable mapping. *E* is called a $C^k -$ differentiable if both E and E^{-1} are $C^k -$ maps.

Stability Theorem: A sufficient condition for a periodic point $\bar{x}$ of period *q* for a diffeomorphism $E: \Re^n \to \Re^n$ to be stable is that the eigenvalues of the derivative $DE^q(\bar{x})$ are less than one in absolute value. Armed with all these ideas and concepts, we now proceed to concentrate to our main aim and objectives

LYAPUNOV EXPONENTS FOR ONE-DIMENSIONAL MAPS

Let us consider a one-dimensional dynamical system $x_{t+1} = \xi(x_t; b), t = 1,2,3,...$., and to determine Lyaponuv exponents formally, we begin by considering an attractor point x_0 and a neighboring attractor point $x_0 + \varepsilon$. We then applying the iterated map function ξ, *n* times to each value and considering

$$D_n \equiv \left|\xi^n(x_0+\varepsilon)-\xi^n(x_0)\right|$$

we expect, if the behavior is chaotic, the above separation to grow exponentially with *n*. So, we may write

$$D_n = \varepsilon e^{\lambda n} \Rightarrow \lambda = \frac{1}{n}\log\left[\frac{\left|\xi^n(x_0+\varepsilon)-\xi^n(x_0)\right|}{\varepsilon}\right]$$

where λ is the Lyaponuv exponent for the trajectory. If we let $\varepsilon \to 0$ and applying the chain rule for differentiation, λ can be put in more intuitive form [52]:

$$\lambda(\textit{the rate of divergence of the two traje ctoties}) = \frac{1}{n}\log\left(\left|\xi'(x_0)\right|.\left|\xi'(x_1)\right|....\left|\xi'(x_{n-1})\right|\right),$$

where $\xi^n(x_0) = \xi'(x_0).\xi'(x_1).....\xi'(x_{n-1})$

This implies $$\lambda = \frac{1}{n}\sum_{i=0}^{n-1}\log\left|\xi'(x_i)\right|$$

The value of the Lyapunov exponent may depend on the initial value. So we may think of average Lyapunov exponent by taking suitable number of points at a time. With the help of a computer program we follow the above procedure to get the Lyapunov exponent.

For a particular value of the parameter *b*, if $x_1^s, x_2^s, ..., x_n^s$ are *n* stable periodic points, then the Lyapunov exponent $\lambda = \frac{1}{n}\sum_{i=0}^{n-1}\log\left|\xi'(x_i)\right|$, where x_0 is the initial value, becomes $\lambda = \sum_{i=1}^{n}\log\left|\xi'(x_i^s)\right|$ when x_0 is sufficiently close to one of $x_i^s, i = 1,2,3,....,n$, since for large value of $n, x_0, x_1, x_3, ..., x_n, ...$ converges to the set $\{x_1^s, x_2^s, ..., x_n^s\}$. This gives $\lambda < 0,$ for

$$\left|\xi'(x_1^s).\xi'(x_2^s).....\xi'(x_n^s)\right| < 1.$$

Thus so long as stable periodic points are there, λ will be negative. However if the normal maxima is a part of attractor for that particular value of the parameter *b*, $\lambda = -\infty$, as $\left|\xi'(x_i^s)\right| = 0$ for some value of *i* where ξ has maximum value. Now suppose

$\hat{b}$ be the bifurcation value and $\hat{b} - \delta$ be a parameter value where 2^n stable periodic point occurs. Then as $\delta \to 0^+$,

$$\xi'(x_1^s).\xi'(x_2^s)......\xi'(x_n^s) = -1 + \sigma,$$

where $$\sigma \to 0^+, x_1^s.x_1^s.....x_r^s, r = 2^n,$$

are 2^n stable periodic points. Therefore,

$$\lambda = \log\left|\xi'(x_1^s).\xi'(x_2^s).....\xi'(x_r^s)\right|$$
$$= \lim_{\sigma\to 0^+}\left|-1+\sigma\right| = 0.$$

Similarly, for $\hat{b}+\delta$ where 2^{n+1} stable periodic point occurs, we have $\lambda = 0.$

So, $$\lim_{b\to\hat{b}}(\lambda) = 0, \hat{b}$$

is a bifurcation value.

LYAPUNOV EXPONENTS FOR OUR MODEL

With the help of the results discussed in 1.3 , we have given the calculated values of the Lyapunov exponents for some values of the parameter *b* of our nonlinear chaotic model:

$$\rho(x) = ax^2 - bx$$

where $x \in [0,4]$, $a = -1$ and $b \in [-1,-4]$ is a tunable parameter. For our calculation we have considered iteration size of 50000 for getting the values in the Table 1.1

Table 1.1

Parameter Values	Lyapunov Exponents	Parameter Values	Lyapunov Exponents
b_1	-0.00020002	-3.56994	-0.00504841
b_2	-0.00020241	**-3.56995**	**0.00314049**
b_3	-0.00005441	-3.56996	0.00583025
b_4	-0.00005671	-3.56998	0.00731921
b_5	-0.00018183	-3.57000	0.0109609
b_6	-0.00014147	-3.57200	0.0503521
b_7	-0.00006973	-3.57400	0.0702857
b_8	-0.00006810	-3.57600	0.0881221

Below we have shown the graph of Lyapunov exponents versus the parameter values between -2.8 to -4.0. The main significance of this figure is that one can easily distinguish the regions which are chaotic (Lyapunov exponent $\lambda > 0$) from regions which tend to a fixed point or a periodic orbit (i.e. $\lambda < 0$). We see several points (the first is at b = -3 where the Lyapunov exponent hits the horizontal line and then goes negative again. These are the period doubling bifurcations. The figure supports the first three bifurcation points as -3.0, -3.44948974278…and -3.54409035955… The Lyapunov exponents calculated at the first eight bifurcations points, that is b_k (k = 1, 2,….,8), are shown in the first column in the Table 1.1, where the Lyaponuv exponent are almost zero. The first chaotic region appears after the parameter value b = -3.56995(approx**).** Moreover, after the first chaotic region, we observe some portions of the graph are in the negative side of the x-axis. They signify that within the chaotic region also, at certain values of the parameter, there are regular behaviors and after that again chaotic region starts. Actually, these are the windows in the chaotic region.

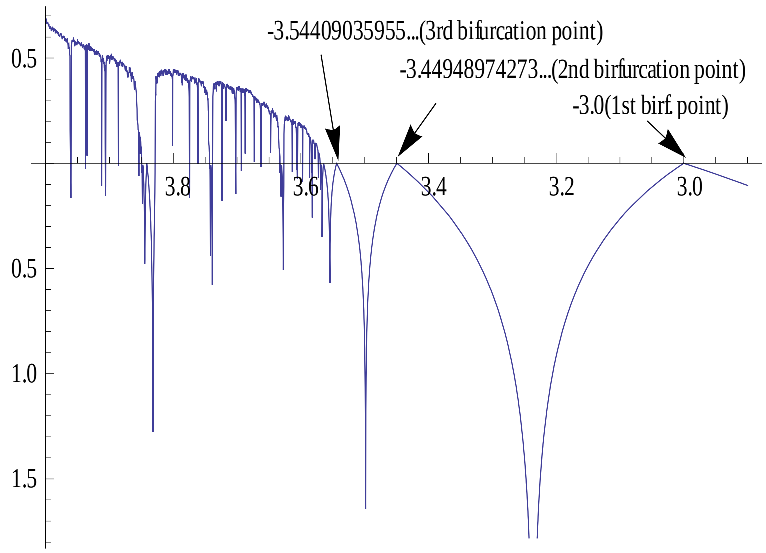


Figure 1.1: Graph of Lyapunov Exponents Versus Parameter B for $-4.0 \le b \le -2.9$

Period-doubling Cascades [3, 4]:

Period	One of the Periodic Points	Bifurcation Points
1	x_1 =2.000000000000…	b_1 = -3.000000000000…
2	x_2 =1.517638090205…	b_2 =-3.449489742783…
4	x_3 =2.905392825125…	b_3 =-3.544090359552…
8	x_4 =3.138826940664…	b_4 =-3.564407266095…
16	x_5 =1.241736888630…	b_5 =-3.568759419544…
32	x_6 =3.178136193507…	b_6 =-3.569691609801…
64	x_7 =3.178152098553…	b_7 =-3.569891259378…
128	x_8 =3.178158223315…	b_8 =-3.569934018374…
256	x_9 =3.178160120824…	b_9 =-3.569943176048…
512	x_{10} =1.696110052289…	b_{10} =-3.569945137342…
1024	x_{11}=1.696240778303…	b_{11} =-3.569945557391…
…	… … …	… … …

Time Series Analysis

In our case, the difference equation is

$$x_{n+1} = -x_n^2 - bx_n , \;\; a = -1 \text{ and } n = 0,1,2,.....$$

On the horizontal axis the number of iterations ('time') is marked, that on the vertical axis the amplitude are given for each iteration. The graphs of time series analysis are exhibited for showing the existence of different periodic orbits of periods 2^k, $k = 0, 1, 2, \ldots$ as well as chaotic behavior. The set towards which the n values converge is called an attractor.

We have seen that an attractor may be a fixed point, a limit cycle, or a chaotic attractor. We look the time series graphs in the following figures.

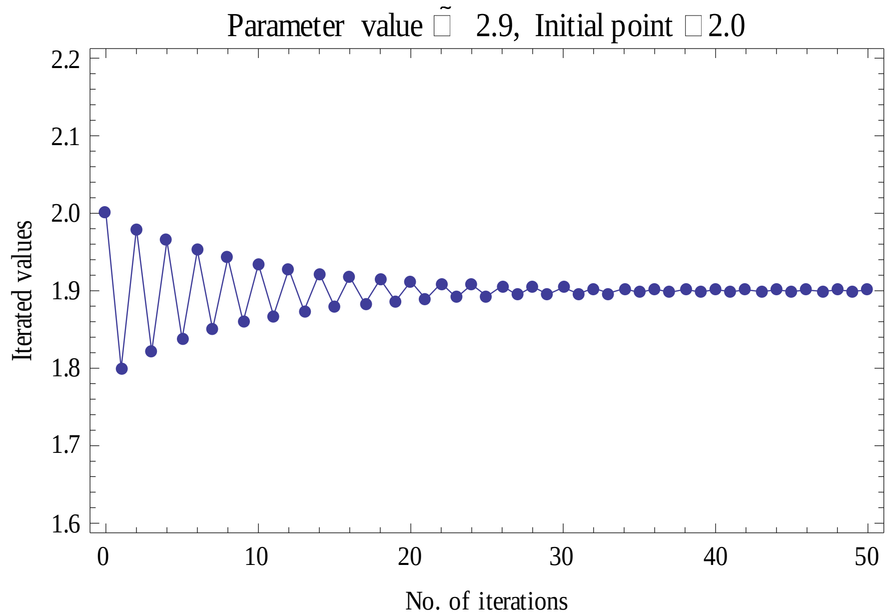


Figure 1.2: The Time Series Showing Period One Behavior

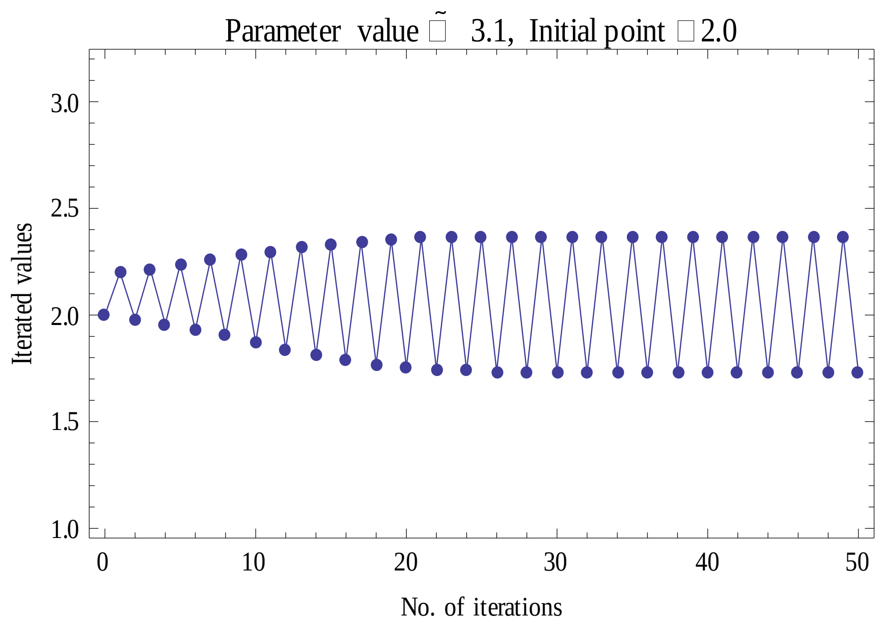
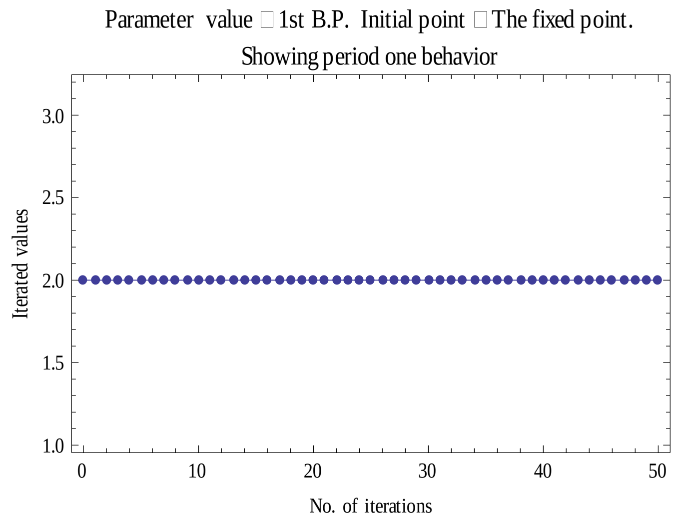
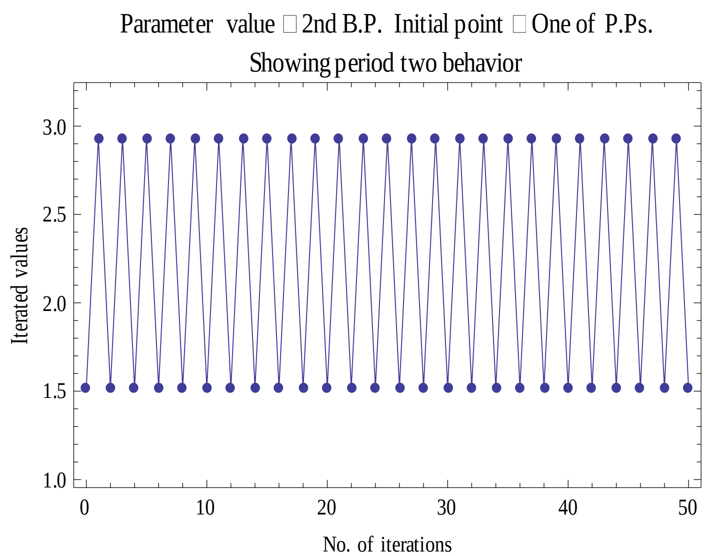


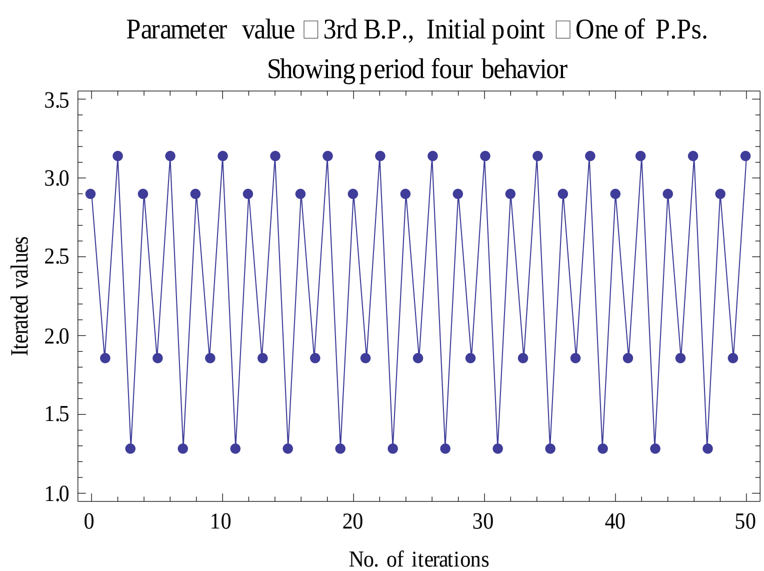
Figure 1.3: The Time Series Showing Period Two Behavior

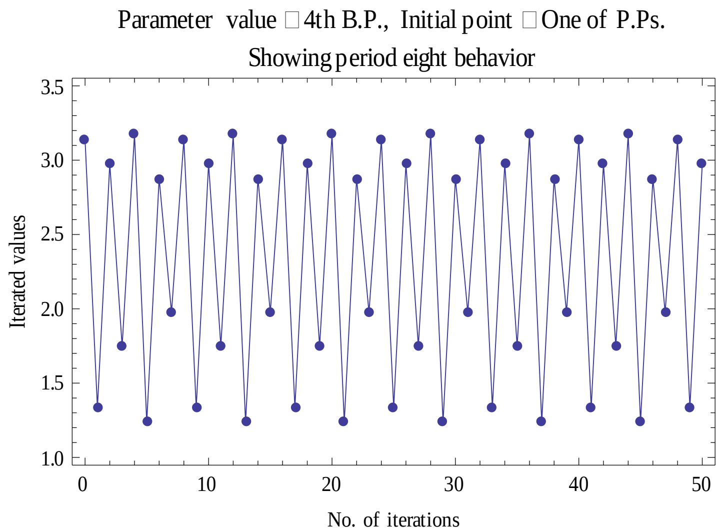
Time Series Graphs

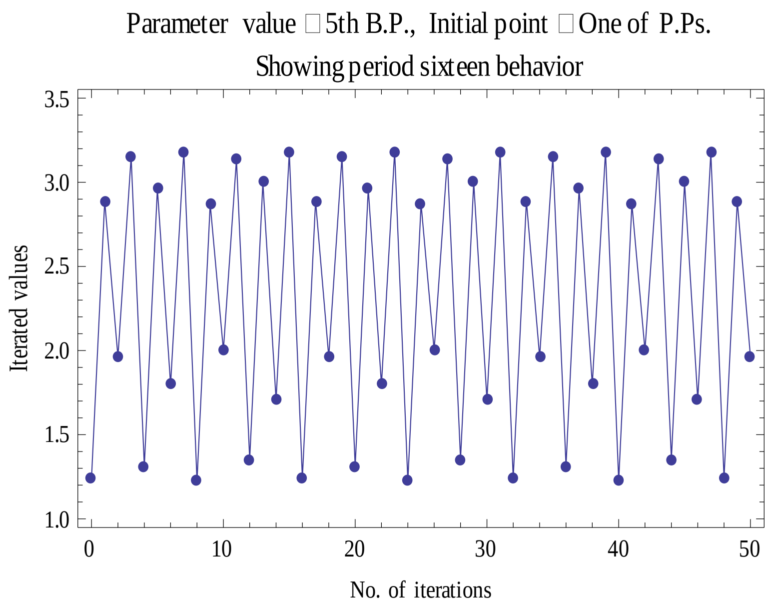
Time series graph at bifurcation points as well as periodic points. In the following figures B.P. and P.Ps. stand for Bifurcation point and Periodic points respectively.

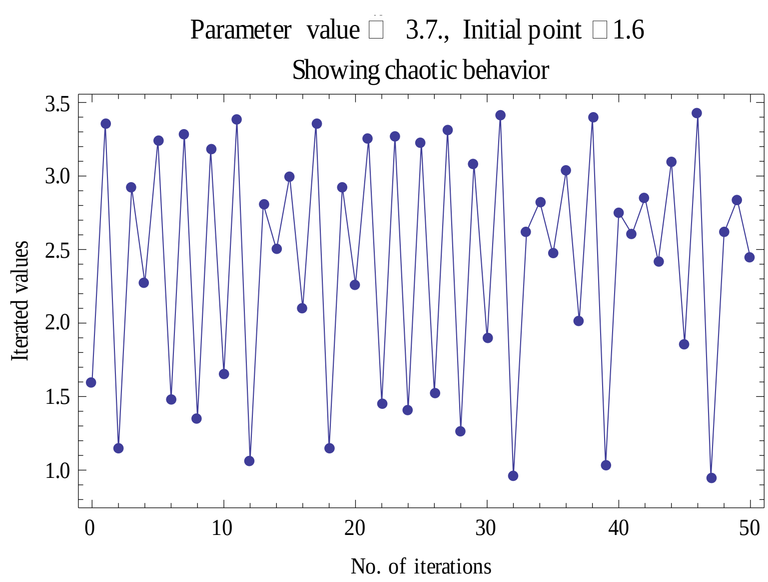

Parameter value □ 1st B.P. Initial point □ The fixed point.
Showing period one behavior
Iterated values
No. of iterations


Parameter value □ 2nd B.P. Initial point □ One of P.Ps.
Showing period two behavior
Iterated values
No. of iterations


Parameter value □ 3rd B.P., Initial point □ One of P.Ps.
Showing period four behavior
Iterated values
No. of iterations


Parameter value □4th B.P., Initial point □One of P.Ps.
Showing period eight behavior
Iterated values
No. of iterations


Parameter value □5th B.P., Initial point □One of P.Ps.
Showing period sixteen behavior
Iterated values
No. of iterations


Parameter value □ 3.7., Initial point □1.6
Showing chaotic behavior
Iterated values
No. of iterations

If we start with a value of b just less than b_1, successive points converges to a fixed point with an initial non-zero value of x. But for values of b slightly greater than b_1 the fixed point 'bifurcates' forming a periodic orbit of period of period-2. This bifurcates again, that is, the period doubles at the large value of b to a periodic orbit of period-4 and so on. In this way, as b increases the period continues to double at successively closer and closer value of b until we get chaotic behavior. This phenomenon can be continued up to the value of b = -3.56994 (approx). After that a chaotic attractor appears.

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