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STATIC SPHERICALLY SYMMETRIC INTERIOR SOLUTIONS WITH NONUNIFORM DENSITY

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ABSTRACT

For the highly nonlinearity conditions, it is so much difficult to obtain exact solution of Einstein's field equations. Many authors have been working on the investigation of exact solution of Einstein's equation. One of these solutions, Schwarzschild uniform density solution is unphysical. In this paper, we have applied the Adler method to obtain Einstein's equations solution in use of different equation of state. We got a new solution although it is Tolman's solution no IV. But our solution is different and original. We have described the obtain solution in terms of two new variables. Also we have shown that maximum mass of 1/3 the fluid radius (in geometric units) which is less than Adler's 2/5 or Schwarzschild 4/9.

KEYWORDS: Non-Uniform Density, **E**xact Solution, Equation of State, Uniform Density

INTRODUCTION

Uniform density solution [1] is unphysical because velocity of sound becomes infinite if $\rho =$ constant. A small number of solutions by authors [2, 3, 4, 5] with non-uniform density distribution which have been obtained are valuable and interesting because they provide some insight into the effects of relativity on the interior structures of stars. In this paper, one such solution is studied. In section-2, the solution found by Adler [2] is reviewed. There is a misprint in equation (3.3d) of Adler's paper which we have corrected in this section. Some interesting properties of Adler's solution are studied in section 3. In section 4 we have shown our new solution by using the method found by Adler. This is a new solution but no different from Tolman's [3] solution no IV. Finally some interesting properties of the solution are studied and defined the results in two new variables x and y in section 5. For this solution we have shown that for a finite radius (r_0) of the fluid sphere $\frac{m_0}{r_0} < \frac{1}{3}$, where m_0 is the mass of the fluid sphere. The star collapses if $\frac{m_0}{r_0}$ exceeds $\frac{4}{9}$.

Buchdahl [4] has shown that the condition $\frac{m_0}{r_0} \le \frac{4}{9}$ is true in general.

ADLER'S STATIC SPHERICALLY SYMMETRIC INTERIOR SOLUTION

Let us take the static spherically symmetric line element in the following form,

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{G^2(r)} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \quad (1)$$

Einstein equation without cosmological constant is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad (2)$$

For a perfect fluid energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu} \tag{3}$$

For the metric (1), equation (2) gives

$$8\pi\rho = -\frac{1}{r^2}\left(r\frac{dG^2}{dr} + G^2 - 1\right) \tag{4}$$

$$8\pi p = \frac{1}{Nr^2}\left(2rG^2\frac{dN}{dr} - N + NG^2\right) \tag{5}$$

$$8\pi p = \frac{1}{Nr}\left(G^2\frac{dN}{dr} + \frac{1}{2}N\frac{dG^2}{dr} + rG^2\frac{d^2N}{dr^2} + \frac{r}{2}\frac{dN}{dr}\frac{dG^2}{dr}\right) \tag{6}$$

From (5) and (6) we obtain

$$\frac{dG^2}{dr} - \frac{2(N + rN' - r^2N'')G^2}{r(N + r\,N')} = -\frac{2N}{r(N + r\,N')}$$

$$\text{or, } \frac{dT}{dr} - P(r)T = -Q(r) \tag{7}$$

where

$$T(r) = G^2(r),$$

$$P(r) = \frac{2(N + rN' - r^2N'')}{r(N + r\,N')} \tag{8}$$

$$\text{and } Q(r) = \frac{2N}{r(N + r\,N')} \tag{9}$$

Solution of (7) becomes simple if we put P(r) = Q(r).

Now the P(r) and Q(r) are the equal if

$$rN' - r^2N'' = 0$$

$$\text{or, } \frac{N''}{N'} = \frac{1}{r}$$

Integrating this we obtain,

$$\log(N') = \log r + \log(2B)$$

$$\text{or, } N' = 2Br \tag{10}$$

Again integrating it we get

$$N = A + Br^2 \tag{11}$$

where A and B are constants.

$$\therefore \text{I.F.of } (7) = e^{-\int P(r)\,dr}$$

Therefore, the solution of (7) is

$$Te^{-\int P(r)dr} = \int Q(r)e^{-\int P(r)dr}dr$$

$$\text{or, } T = 1 + Ce^{\int P(r)\,dr}dr \tag{12}$$

Now, $P(r) = \dfrac{2(A+Br^2)}{r(A+3Br^2)}$ [putting the value of N from (11)]

$$\therefore \int P(r)dr = \int (\frac{2}{r} - \frac{4Br)}{A+3Br^2})dr$$

$$= 2\log r - \frac{2}{3}\log(A + 3Br^2)$$

$$\therefore e^{\int P(r)dr} = \frac{r^2}{(A+3Br^2)^{\frac{2}{3}}}$$

So from (12) we get

$$T = 1 + C \cdot \frac{r^2}{(A + 3Br^2)^{\frac{2}{3}}}$$

$$\text{i.e, } G^2 = 1 + \frac{Cr^2}{(A + 3Br^2)^{\frac{2}{3}}} \tag{13}$$

Sometimes it is convenient to replace G(r) by another function m(r) defined by

$$G^2(r) = 1 - \frac{2m(r)}{r}$$

So from (13) we get

$$1 + \frac{Cr^2}{(A+3Br^2)^{\frac{2}{3}}} = 1 - \frac{2m(r)}{r}$$

$$\text{or, } m(r) = 1 + \frac{Cr^3}{2(A + 3Br^2)^{\frac{2}{3}}} \tag{14}$$

If r_0 is the radius of the fluid sphere and m_0 its mass, then we have

$$\varepsilon = \frac{m_0}{r_0} = -\frac{Cr_0^{\ 2}}{2(A + 3Br_0^{\ 2})^{\frac{2}{3}}} \tag{15}$$

From (5) we get

$$8\pi p = \frac{2}{r} \cdot \frac{2Br}{A + Br^2} \cdot G^2 - \frac{1}{r^2}(1 - G^2)$$

On the surface of the sphere p = 0, $r = r_0$.

This implies

$$\frac{Br_0^{\ 2}}{A + Br_0^{\ 2}}(1 - 2\varepsilon) = 2\varepsilon$$

$$\text{or, } \varepsilon A + r_0^{\ 2}(5\varepsilon - 2)B = 0 \tag{16}$$

Also we have by matching with the Schwarzschild exterior solution

$$A + B r_0^{\ 2} = \sqrt{1 - 2\varepsilon} \tag{17}$$

$$\text{or, } \varepsilon A + \varepsilon B r_0^{\ 2} = \varepsilon\sqrt{1 - 2\varepsilon} \tag{18}$$

Subtracting (16) from (18) we get

$$B = \frac{\varepsilon}{2r_0^{\ 2}\sqrt{(1 - 2\varepsilon)}} \tag{19}$$

Putting this value of B in (17) we get

$$A = \frac{2 - 5\varepsilon}{2\sqrt{(1 - 2\varepsilon)}} \tag{20}$$

From (15) we get

$$C = -\frac{2\varepsilon}{r_0^{\ 2}} \frac{(1 - \varepsilon)^{\frac{2}{3}}}{(1 - 2\varepsilon)^{\frac{1}{3}}} \tag{21}$$

Now from (11)

$$N = \frac{2 - 5\varepsilon}{2\sqrt{(1 - 2\varepsilon)}} + \frac{\varepsilon}{2\sqrt{(1 - 2\varepsilon)}}\left(\frac{r}{r_0}\right)^2$$

$$\text{or, } N = \left(1 - \frac{5}{2}\varepsilon + \frac{1}{2}\varepsilon y^2\right)(1 - 2\varepsilon)^{-\frac{1}{2}} \tag{22}$$

Where $y = \frac{r}{r_0}$.

From (13) we get

$$G^2 = 1 - \frac{2\varepsilon(1-\varepsilon)^{\frac{2}{3}} y^2}{(1 - \frac{5}{2}\varepsilon + \frac{1}{2}\varepsilon y^2)^{\frac{2}{3}}} \tag{23}$$

Also from (14) we get

$$m(r) = \frac{m_0 (1-\varepsilon)^{\frac{2}{3}} y^3}{(1 - \frac{5}{2}\varepsilon + \frac{1}{2}\varepsilon y^2)^{\frac{2}{3}}} \tag{24}$$

From (5) pressure p(r) is given by

$$p(r) = \frac{\varepsilon}{4\pi r_0^2}[G^2(1 - \frac{5}{2}\varepsilon + \frac{1}{2}\varepsilon y^2)^{-1} - (1-\varepsilon)^{\frac{2}{3}}(1 - \frac{5}{2}\varepsilon + \frac{3}{2}\varepsilon y^2)^{-\frac{2}{3}}] \tag{25}$$

There is a misprint in equation (3.3d) of Adler's paper which has been correct in equation (25).

Similarly from (4), density distribution is given by

$$\rho(r) = \frac{\varepsilon}{4\pi r_0^2}\{\frac{(1-\varepsilon)^{\frac{2}{3}}}{(1 - \frac{5}{2}\varepsilon + \frac{3}{2}\varepsilon y^2)^{\frac{2}{3}}}\}\{3 - \frac{2\varepsilon y^2}{1 - \frac{5}{2}\varepsilon + \frac{3}{2}\varepsilon y^2}\} \tag{26}$$

PROPERTIES OF THE SOLUTION

Central density ρ_c is obtained by putting y = 0 in (26). We get

$$\rho_c = \frac{3\varepsilon}{4\pi r_0^2}\left(\frac{1-\varepsilon}{1 - \frac{5}{2}\varepsilon}\right)^{\frac{2}{3}} \tag{27}$$

We see from (27) that the central density ρ_c is a function of $\varepsilon = \frac{m_0}{r_0}$. Clearly for a fixed r_0, the mass m_0 is a function of ρ_0. m_0 is maximum if

$$\frac{dm_0}{d\rho_c} = 0$$

$$\Rightarrow \frac{d(\frac{m_0}{r_0})}{d\rho_c} = 0 \quad \text{[Since by assumption } r_0 = \text{constant]}$$

$$\Rightarrow \frac{d\varepsilon}{d\rho_c} = 0$$

$$\Rightarrow \frac{d\rho_c}{d\varepsilon} = \infty \qquad (28)$$

From (28) we get

$$\varepsilon = \frac{2}{5} \quad \Rightarrow m_0 = \frac{2}{5} r_0$$

Thus for a fixed r_0 maximum is given by $m_0 = \frac{2}{5} r_0$. This may be compared with the maximum mass $m_0 = \frac{1}{2} r_0$ obtained in the Schwarzschild interior solution by demanding that the Schwarzschild radius not be exterior to be fluid sphere. It should be noted that $\frac{2}{5} r_0 < \frac{1}{2} r_0$. In the solution found by Adler we see that m_0 cannot exceed $\frac{2}{5} r_0$ whereas in the Schwarzschild interior solution m_0 is bounded from above by $m_0 \leq \frac{1}{2} r_0$.

Central pressure p_c is obtained by putting y = 0 in (25). We get

$$p_c = \frac{\varepsilon}{4\pi r_0^2} \left[\frac{1}{1 - \frac{5}{2}\varepsilon} - \left(\frac{1 - \varepsilon}{1 - \frac{5}{2}\varepsilon} \right)^{\frac{2}{3}} \right] \qquad (29)$$

Now p_c becomes infinite if $\varepsilon = \frac{2}{5} \quad \Rightarrow m_0 = \frac{2}{5} r_0$

Hence $m_0 < \frac{2}{5} r_0$

Schwarzschild interior solution has the similar property that p_c becomes infinite when $\varepsilon = \frac{4}{9}$ which implies $m_0 < \frac{4}{9} r_0$.

A NEW SOLUTION [TOLMAN'S SOLUTION NO. IV]

In this section we have shown that a new solution derived by using the Adler's method. It is nothing but Tolman's solution no IV.

In (7) let us put $P(r) = \frac{3}{r} + \frac{Q'}{Q}$. Then we get

$$\frac{2N + 2rN' - 2r^2 N''}{r(N + r\,N')} = \frac{3}{r} + \frac{N'}{N} - \frac{N + 3rN' + r^2 N''}{r(N + r\,N')}$$

$$\text{or, } r^2 N'^2 + r^2 NN'' - rNN' = 0 \qquad (30)$$

Put $\alpha = N^2$ in (30), then

$$\alpha' = 2NN', \; \alpha'' = 2N'^2 + 2NN''$$

Thus (30) becomes

$$r^2\alpha'' - r\alpha' = 0$$

or, $$\frac{\alpha''}{\alpha'} = \frac{1}{r}$$

$\Rightarrow \alpha = ar^2 + b$, where a and b are integrating constants.

$$\Rightarrow N^2 = ar^2 + b \tag{31}$$

$$\therefore N = \sqrt{ar^2 + b}$$

Now integrating factor of (7) is given by

$$\text{I.F} = e^{-\int P\,dr}$$

$$= e^{-\ln(r^3Q)}$$

$$= \frac{1}{r^3Q}$$

Therefore, the solution of (7) is

$$T \times \frac{1}{r^3Q} = -\int Q \times \frac{1}{r^3Q}dr$$

$$= -\int r^{-3}dr$$

$= \frac{1}{2r^2} + d$, where d is a constant of integration.

$$= \frac{2dr^2 + 1}{2r^2}$$

$$\therefore T = \frac{2dr^2 + 1}{2r^2} \times r^3Q$$

$$= \frac{Qr(2dr^2 + 1)}{2}$$

From (9) we get

$$Q(r) = \frac{2\sqrt{ar^2 + b}}{r(\sqrt{ar^2 + b} + \frac{2ar^2}{2\sqrt{ar^2 + b}})}$$

or, $$rQ(r) = \frac{2(ar^2 + b)}{(2ar^2 + b)}$$

$$\therefore T = \frac{2(ar^2+b)}{2ar^2+b}\cdot\frac{2dr^2+1}{2}$$

$$= \frac{(\frac{a}{b}r^2+1)(1-cr^2)}{2\frac{a}{b}r^2+1}\text{, where } -c = 2d$$

$$\text{or, } G^2(r) = \frac{(\frac{a}{b}r^2+1)(1-cr^2)}{2\frac{a}{b}r^2+1} \tag{32}$$

$$\therefore \frac{1}{G^2(r)} = \frac{2\frac{a}{b}r^2+1}{(\frac{a}{b}r^2+1)(1-cr^2)}$$

For this solution ρ and p are given by

$$8\pi\rho = \frac{a}{b}\cdot\frac{1+\frac{3bc}{a}+3cr^2}{1+2\frac{a}{b}r^2}+\frac{2a}{b}\cdot\frac{1-cr^2}{(1+2\frac{a}{b}r^2)^2} \tag{33}$$

$$8\pi p = \frac{a}{b}\cdot\frac{1-\frac{bc}{a}-3cr^2}{1+2\frac{a}{b}r^2} \tag{34}$$

CONCLUDING PROPERTIES OF THE SOLUTION

Central density ρ_c and central pressure p_c can be obtained by putting r = 0 in (33) and (34) respectively. We obtain

$$\rho_c = \frac{1}{8\pi}\{\frac{a}{b}(1+\frac{3bc}{a})+\frac{2a}{b}\}$$

$$\text{or, } \rho_c = \frac{3}{8\pi}(\frac{a}{b}+c) \tag{35}$$

Similarly

$$p_c = \frac{1}{8\pi}(\frac{a}{b}-c) \tag{36}$$

Let r_0 be the radius of the fluid sphere. Then p = 0 when r = r_0. This implies

$$r_0^2 = \frac{1}{3c}(1-\frac{bc}{a}) \tag{37}$$

From (35) and (36) we get

$$\frac{a}{b} = \frac{4\pi}{3}(\rho_c + 3p_c) \tag{38}$$

$$c = \frac{4\pi}{3}(\rho_c - 3p_c) \tag{39}$$

Putting the values of $\frac{a}{b}$ and c in (37) we get

$$r_0^2 = \frac{3}{2\pi(\frac{\rho_c}{p_c}+3)(\frac{\rho_c}{p_c}-3)} \tag{40}$$

From (40) we see that r_0 depends only on $\frac{\rho_c}{p_c}$. It is also clear that for a finite radius we must have $\frac{\rho_c}{p_c} > 3$.

Let m_0 be the mass of the fluid sphere. Matching with the Schwarzschild solution, then gives

$$1 - \frac{2m_0}{r_0} = \frac{(\frac{a}{b}r_0^2 + 1)(1 - cr_0^2)}{2\frac{a}{b}r_0^2 + 1}$$

or, $$\frac{m_0}{r_0} = \frac{1}{2}[1 - \frac{(\frac{a}{b}r_0^2 + 1)(1 - cr_0^2)}{2\frac{a}{b}r_0^2 + 1}] \tag{41}$$

Putting the values of $\frac{a}{b}$, c and r_0^2 from (38), (39) and (40) in (41) we get

$$\frac{m_o}{r_o} = \frac{2}{\frac{\rho_c}{p_c} + 3} \tag{42}$$

From (42) we see that $\frac{m_o}{r_o}$ depends only on $\frac{\rho_c}{p_c}$. Clearly $\frac{m_o}{r_o} < \frac{1}{2}$ if $\rho_c > p_c$, $\frac{m_o}{r_o} = \frac{1}{2}$ if $\rho_c = p_c$ and $\frac{1}{2} < \frac{m_o}{r_o} < \frac{2}{3}$ if $\rho_c > p_c$. Let us put $y = \frac{m_o}{r_o}$ and $x = \frac{1}{\frac{\rho_c}{p_c} + 3}$ in (42). Then we get the equation

$$y = 2x \tag{43}$$

The graph of (43) is a straight line passing through the origin.

If we assume that $\frac{\rho_c}{p_c}$ is positive then the range of x is given by $0 < x < \frac{1}{3}$. But from (40) we have $\frac{\rho_c}{p_c} < 3$. This implies $x < \frac{1}{6}$. Hence the range of x is further limited to $0 < x < \frac{1}{6}$. In $0 < x < \frac{1}{6}$ for a fixed ρ_c we have $0 < p_c < \frac{\rho_c}{3}$.

Thus we have the following results:

$$p_c < \frac{\rho_c}{3} \text{ and } \frac{m_o}{r_o} < \frac{1}{3}.$$

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