QUALITATIVE PROPERTIES OF SOLUTION OF CROSS-DIFFUSION MODEL OF KOLMOGOROV-FISHER TYPE BIOLOGICAL POPULATION TASK

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ABSTRACT

In present work explores the issues of global solvability of the biological population task of Kolmogorov-Fisher type and qualitative properties of the solution of task on the basis of self-similar analysis. Considered a parabolic system of two quasilinear equations of reaction-diffusion. Suggested suitable initial approximation for fast convergence iterative process. Carried out numerical experiments with visualization for different values of system parameters. Modeling of growth processes of dissipative structures in reaction-diffusion (RD) systems contributes to the development of theoretical ideas about colonial organization of populations.

KEYWORDS: Cross-Diffusion, Biological Population, Parabolic System of Quasilinear Equations, Initial Approximation, Numerical Solution of the Iterative Process, Self-Similar Solutions

1. INTRODUCTION

Let's consider in the domain \( Q = \{(t,x): 0 < t < \infty, \; x \in \mathbb{R}^2 \} \) parabolic system of two quasilinear reaction-diffusion equations of Kolmogorov-Fisher type biological population task

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial x} \left[ (a_{11}u_1^n + a_{12}u_2^n) \frac{\partial u_1}{\partial x} + (b_{11}u_1^n + b_{12}u_2^n) \frac{\partial u_2}{\partial x} \right] + k_1(t)u_1(1 - u_2^\beta_1) \\
\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial x} \left[ (a_{21}u_1^n + a_{22}u_2^n) \frac{\partial u_1}{\partial x} + (b_{21}u_1^n + b_{22}u_2^n) \frac{\partial u_2}{\partial x} \right] + k_2(t)u_2(1 - u_1^\beta_2)
\end{align*}
\]

(1)

\( a_{ij}, \; b_{ij} \) - positive real numbers, \( \beta_1, \beta_2 \geq 0 \), \( u_1 = u_1(t,x) \geq 0 \), \( u_2 = u_2(t,x) \geq 0 \) - desired solutions.

At \( a_{ij} \neq 0 \), \( b_{ij} = 0 \) or \( a_{ij} = 0 \), \( b_{ij} \neq 0 \) mathematical model (1) is a system of reaction-diffusion with diffusion coefficients \( a_{ij}u_1^n \geq 0 \), \( b_{ij}u_2^n \geq 0 \). In the case when at least one of the coefficients \( a_{ij} \neq 0 \) or \( b_{ij} \neq 0 \) (sign can be any), system (1) is the cross-diffusion (mutual-diffusion for \( i,j=1,2 \)) [1-12].

Equation (1) is a generalization of the simple diffusion model for the logistic model of Malthusian type population growth \([13-16]\) \( (f_1(u_1, u_2) = u_1, \; f_1(u_1, u_2) = u_2, \; f_2(u_1, u_2) = u_1, \; f_2(u_1, u_2) = u_2) \). Ferhulst type \( (f_1(u_1, u_2) = u_1(1 - u_2), \; f_1(u_1, u_2) = u_2(1 - u_1), \; f_2(u_1, u_2) = u_1(1 - u_2), \; f_2(u_1, u_2) = u_2(1 - u_1)) \), and Allee type \( (f_1(u_1, u_2) = u_1(1 - u_2^{\beta_1}), \; f_1(u_1, u_2) = u_2(1 - u_1^{\beta_1}), \; f_2(u_1, u_2) = u_1(1 - u_2^{\beta_2}), \; f_2(u_1, u_2) = u_2(1 - u_1^{\beta_2})) \), \( \beta_1 > 1, \; \beta_2 > 1 \) for the case of dual nonlinear diffusion. When \( \beta_1 \geq 1, \; \beta_2 \geq 1 \), it can also be considered as the equation
of nonlinear filtering, thermal conductivity with simultaneous exposure of the source and the absorption capacity of which is equal respectively $u_1, -u_2^\beta, u_2, -u_2^\beta$, under the influence of convective transfer with speeds $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ in a two-component nonlinear media [15,19].

Let’s consider a spatial analogue of the system of Volterra-Lotka with nonlinear power dependence of diffusion coefficient on population density. In the simplest case walterovich competitive interactions between populations can be constructed numerically, and in some cases analytically heterogeneous in space solutions [19]. In dissertation work "Cross diffusion systems" Toan Trong Nguyen, M.S. 2006 reviewed some qualitative properties of solutions of problem (1), but the solution is not obtained. The aim of this work is the investigation of qualitative properties of solutions of problem (1) based on the self-similar analysis and numerical solutions using modern computer technologies, research methods linearization to the convergence of iterative process with further visualization. Researched global solvability of problem (1) based on the self-similar approach.

Let's construct a similar system of equations (1) - easier to research the system of equations. Construct it by using nonlinear splitting [15], which will make the following substitution in (1)

$$u_1(t,x) = e^{-\int^{\xi(\tau)}_{0}k_1(\zeta)d\zeta}v_1(\tau(t),x),$$

$$u_2(t,x) = e^{-\int^{\xi(\tau)}_{0}k_1(\zeta)d\zeta}v_2(\tau(t),x),$$

leading (1) to the form

$$\frac{\partial v_1}{\partial \tau_1} = \frac{\partial}{\partial x}\left[(a_{11}v_1^m + a_{12}v_2^m)\frac{\partial v_1}{\partial x} + (b_{11}v_1^m + b_{12}v_2^m)\frac{\partial v_2}{\partial x}\right] + k_1(t)\cdot e^{(\beta_1-k_1)m\tau_1}v_1^\beta v_2^\beta,$$

$$\frac{\partial v_2}{\partial \tau_2} = \frac{\partial}{\partial x}\left[(a_{21}v_1^m + a_{22}v_2^m)\frac{\partial v_1}{\partial x} + (b_{21}v_1^m + b_{22}v_2^m)\frac{\partial v_2}{\partial x}\right] + k_2(t)\cdot e^{(\beta_2-k_2)m\tau_2}v_1^\beta v_2^\beta,$$

$$v_1\big|_{t=0} = v_{10}(x), \quad v_2\big|_{t=0} = v_{20}(x).$$

By selecting $\tau = \tau_1 = \tau_2 = \frac{e^{\text{ext}}}{km}$, we get the following system of equations

$$\frac{\partial v_1}{\partial \tau} = \frac{\partial}{\partial x}\left[(a_{11}v_1^m + a_{12}v_2^m)\frac{\partial v_1}{\partial x} + (b_{11}v_1^m + b_{12}v_2^m)\frac{\partial v_2}{\partial x}\right] + a_1(t)\tau^h v_1 v_2^\beta,$$

$$\frac{\partial v_2}{\partial \tau} = \frac{\partial}{\partial x}\left[(a_{21}v_1^m + a_{22}v_2^m)\frac{\partial v_1}{\partial x} + (b_{21}v_1^m + b_{22}v_2^m)\frac{\partial v_2}{\partial x}\right] + a_2(t)\tau^h v_1^\beta v_2,$$

Where $b_i = \frac{\beta_i - m}{m}; \quad a_i = k_i(t)\cdot \lfloor km \rfloor^h; \quad b_2 = \frac{\beta_2 - m}{m}; \quad a_2 = k_2(t)\cdot \lfloor km \rfloor^h.$

If $b_i = 0$, and $a_i(t) = \text{const}, \quad i = 1, 2$, the system has the form
In this case, on the basis of the method of splitting [19] are looking for a solution in the form
\[ v_1(t,x) = v_1(\tau(w(t,x)),x), w_1(\tau(t),x) = f_1(\xi), \]
\[ v_2(t,x) = v_2(\tau(w(t,x)),x), w_2(\tau(t),x) = f_2(\xi), \quad \xi = \frac{x}{\sqrt{\tau_1}} \]

Where
\[ \mathcal{V}_1(\tau) = (T_0 + \tau)^{-\gamma_1}, \quad \mathcal{V}_2(\tau) = (T_0 + \tau)^{-\gamma_2}, \quad T_0 > 0, \quad \gamma_1 = \frac{1}{\beta_1}, \quad \gamma_2 = \frac{1}{\beta_2}. \]

And in the case \( \beta_i \neq 0, \) and \( a_i(t) = \text{const}, i = 1,2 \)
\[ \mathcal{V}_1(\tau) = (T_0 + \tau)^{-\gamma_1}, \quad \mathcal{V}_2(\tau) = (T_0 + \tau)^{-\gamma_2}, \quad T_0 > 0, \]

Where
\[ \gamma_1 = \frac{b_2 + 1}{\beta_2}; \quad \gamma_2 = \frac{b_1 + 1}{\beta_1}. \]

Then we then obtain a self-similar system
\[
\begin{align*}
\frac{d}{d\xi} \left[ (a_{11}f_1^{w_1} + a_{12}f_2^{w_2}) \frac{df_1}{d\xi} + (b_{11}f_1^{w_1} + b_{12}f_2^{w_2}) \frac{df_2}{d\xi} \right] + \frac{\xi}{2} \frac{df_1}{d\xi} + \psi_1 f_1 (1 - f_1^{b_1}) = 0, \\
\frac{d}{d\xi} \left[ (a_{21}f_1^{w_1} + a_{22}f_2^{w_2}) \frac{df_1}{d\xi} + (b_{21}f_1^{w_1} + b_{22}f_2^{w_2}) \frac{df_2}{d\xi} \right] + \frac{\xi}{2} \frac{df_2}{d\xi} + \psi_2 f_2 (1 - f_2^{b_2}) = 0, 
\end{align*}
\]

And \( \tau_1 = \tau_1(t) \) is chosen so
\[
\tau_1(t) = \begin{cases} 
(T + \tau)^{-\gamma_m+1} - \gamma_m, & \text{if} - \gamma_m+1 \neq 0, \\
\ln(T + \tau), & \text{if} - \gamma_m+1 = 0, \\
(T + \tau), & \text{if} m = 0, 
\end{cases}
\]

If \( \gamma_2 m_1 = \gamma_1 m_2 \)

System (5) has an approximate solution in the form
\[ f_1 = A(a - b_1 \xi^2)^{m_1}, \quad f_2 = B(a - b_2 \xi^2)^{m_2}, \quad (y)_{\max} = \max(0, y). \]
Investigation of qualitative properties of system (1) is allowed to perform a numerical experiment based on the values included in the system of numeric parameters. For this purpose, as the initial approximation was used asymptotic solutions. For the numerical solution of the problem for the linearization of system (1) has been used linearization methods of Newton and Picard. To construct a similar system of equations biological populations used method for nonlinear splitting [16, 19]

On the basis of research numerical experiments in

Below are the results of numerical experiments for different parameter values obtained in Mathcad.

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<th>$t = 1$</th>
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Figure 1: Results of Numerical Experiment
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