International Journal of General Engineering and Technology (IJGET)
ISSN(P): 2278-9928; ISSN(E): 2278-9936
Vol. 5, Issue 3, Apr – May 2016, 1-6

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations
IASET

ON THE DERIVATIVE OF SLOW GROWTH OF ANALYTIC FUNCTIONS IN THE HALF PLANE

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ABSTRACT

If f(s) = $\sum_{n=1}^{\infty} a_n\, e^{s\lambda n}$ be a Dirichlet series, where s = σ+ it. $f^{(n)}(s)$ denote the derivative of $f^{(s)}$. And it is also analytic for σ <α. The growth of analytic function f(s) is studied through the order, lower order and type etc. Using the results we find, the relative growth of $m(\sigma_1 f^{(n)})$ and $N(\sigma_1 f^{(n)})$ with respect to $m(\sigma_1 f)$ and $N(\sigma_1 f)$, where $m(\sigma_1 f)$ and $N(\sigma_1 f)$ are known as maximum term and rank of maximum term.

KEYWORDS: Dirichlet Series, Analytic Function, Derivative Order, Lower Order

INTRODUCTION

Let f(s) = $\sum_{n=1}^{\infty} a_n\, e^{s\lambda n}$ be a Dirichlet series, where $0 < \lambda_1 < \lambda_2 < ..... < \lambda_n \to \infty$ as $n \to \infty$, s = σ+ it (σ and t are real) and $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We assume that

$$\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \delta > 0 \tag{1.1}$$

Then we also have

$$\lim_{n\to\infty} Sup \left(\frac{n}{\lambda_n}\right) = D < \infty \tag{1.2}$$

The Dirichlet series defined above represents an analytic function in the half plane σ < α [1], [4]

$$\alpha = \lim_{n\to\infty} Sup \frac{\log|a_n|^{-1}}{\lambda_n} \tag{1.3}$$

For σ <α, M (σ), M (σ) and N (σ) are defined as follows:

$$M(\sigma) = \underset{-\infty<t<\infty}{lub}|f(\sigma + it)|,$$

$$M(\sigma) = \underset{n\geq 1}{max}\{|a_n|e^{\sigma\lambda n}\},$$

$$\text{And } N(\sigma) = \underset{n\geq 1}{max}\{n : m(\sigma) = |a_n|e^{\sigma\lambda n}\},$$

Let $f^{(n)}(s)$ denote the n^{th} derivative of $f^{(s)}$. Then $f^{(n)}(s)$ is also analytic for σ < α. Likewise, $M(\sigma_1 f^{(n)})$, $m(\sigma_1 f^{(n)})$, $n(\sigma_1 f^{(n)})$ can also be defined for the derivative $f^{(n)}(s)$.

The growth of analytic function f(s) is studied through the order, lower order and type etc. The order P and lower orderλ, of f(s) are defined as [2]

$$\begin{matrix}\lim sup \\ \sigma\to\alpha\ inf\end{matrix} \frac{\log\log M(\sigma)}{\log(1-e^{\sigma-\alpha})^{-1}} = \frac{p}{\lambda}\ (0 \le \lambda \le P \le \infty) \quad (1.4)$$

The study the growth of analytic function f(s) when the order P = 0, the concept of logarithmic order P* and lower logarithmic order λ^* of f(s) introduced and they are defined as [3]

$$\begin{matrix}\lim sup \\ \sigma\to\alpha\ inf\end{matrix} \frac{\log\log M(\sigma)}{\log\log(1-e^{\sigma-\alpha})^{-1}} = \frac{P^*}{\lambda^*}\ (1 < \lambda^* \le P^* < \infty) \quad (1.5)$$

It has also been proved that

$$\begin{matrix}\lim sup \\ \sigma\to\alpha\ inf\end{matrix} \frac{\log\log M(\sigma)}{\log\log(1-e^{\sigma-\alpha})^{-1}} = \frac{P^*}{\lambda^*} \quad (1.6)$$

And

$$P^* - 1 \le \lim_{\sigma\to\alpha} \text{Sup} \frac{\log[\lambda_{N(\sigma)}(1-e^{\sigma-\alpha})]}{\log\log(1-e^{\sigma-\alpha})^{-1}} \le P^* \quad (1.7)$$

The coefficient characterization of logarithmic order P^* have been obtained under the stronger condition as (1.1)

$$\text{Max}\left\{1, \lim_{\sigma\to\infty}\sup \frac{\log^+(\alpha\lambda_n + \log|a_n|)}{\log\log\lambda_n}\right\} = P^* \quad (1.8)$$

In this chapter we study the relative growth of $m(\sigma_1 f^{(n)})$ and $N(\sigma_1 f^{(n)})$ with respect to $m(\sigma_1 f)$ and $N(\sigma_1 f)$.

2. Theorem 1: Let f(s) = $\sum_{n=1}^{\infty} a_n e^{s\lambda n}$ be analytic function in the half plane $\sigma < \alpha$, satisfying (1.3). If logarithmic order of f(s) then

$$\begin{matrix}\lim sup \\ \sigma\to\alpha\ inf\end{matrix} \frac{\log\log M(\sigma_1 f^{(1)})}{\log\log(1-e^{\sigma-\alpha})^{-1}} = \frac{P^*}{\lambda^*}\ (1 < \lambda^* \le P^* < \infty) \quad (2.1)$$

Where $f^{(1)}(s)$ is the derivative of f(s).

Proof we have

$$f^{(1)}(s) = \sum_{n=1}^{\infty} a_n \lambda_n e^{s\lambda n} \quad (2.2)$$

The above series converges absolutely for $\sigma \le \alpha_1 < \alpha$. Hence we have

$$|a_n|\ \lambda_n = \lim_{T\to\infty}\left|\frac{1}{2T}\int_{-T}^{T} e^{-(\sigma+it)\lambda n} f^{(1)}(\sigma+it)\,dt\right|$$

Or $|a_n|\ \lambda_n <$ exp $(-\sigma\lambda_n)$ $M(\sigma_1 f^{(1)})$ for all values of n. Then we have

$$\lambda_{N(\sigma)}\ m(\sigma) < M(\sigma_1 f^{(1)}) \quad (2.3)$$

Or $\frac{\log\log m(\sigma)}{\log\log(1-e^{\sigma-\alpha})^{-1}} + \frac{\log[1+\log\lambda_{N(\sigma)}/\log M(\sigma)}{\log\log(1-e^{\sigma-\alpha})^{-1}}$

$\le \frac{\log\log M(\sigma)}{\log\log(1-e^{\sigma-\alpha})^{-1}}$

Taking limits as $\sigma \to \alpha$ we get (2.1),

To obtain reverse inequality, use (1.8).

For any given $\in > 0$ and all $n > n(\in)$ we have

$\text{Log} |a_n| < (\log \lambda_n)^{\mu} - \alpha \lambda_n, \mu = P^* + \in$.

Also

$$M(\sigma_1 f^{(1)}) \leq \sum_{n=1}^{\infty} \exp[2(\log \lambda_n)^{\mu} + (\sigma - \alpha)\lambda_n] \tag{2.5}$$

Or $M(\sigma_1 f^{(1)}) \leq Q(n_0) + \sum_{n=n_0+1}^{\infty} \exp[2(\log \lambda_n)^{\mu} + (\sigma - \alpha)\lambda_n]$,

Where $Q(n_0)$ is the sum of first n_o terms and is bounded. Now it can easily be seen that if

$H(x) = (\log x)^{\mu} - r_x, \qquad r > 0$

Then $\max_{0 \leq x < \infty} H(x) = \left[\log\left(\frac{\mu}{r}\right)\right]^{\mu} - \mu$

$M(\sigma_1 f^{(1)}) < 0(1) + N \exp\left[2\left(\log \frac{2\mu}{\alpha - \sigma}\right)^{\mu} - \mu\right] +$

$\sum_{n=N+1}^{\infty} \exp[2(\log \lambda_n)^{\mu} - (\alpha - \sigma)\lambda_n]$

Also for σ sufficiently close to α and $n > N$, we have

$\sum_{n=N+1}^{\infty} \exp\{2(\log \lambda_n)^{\mu} - (\alpha - \sigma)\lambda_n\} < \sum_{n=N=1}^{\infty} \left\{\frac{-n(\alpha - \sigma)}{2(D - \epsilon)}\right\}$

Where $\lambda_n > \log\left(\frac{4}{\alpha - \sigma}\right)$ and $\limsup_{n \to \infty} \frac{n}{\lambda_n} = D < \infty$

$\therefore \sum_{n=N+1}^{\infty} \exp\{(2 \log \lambda_n)^{\mu} - (\alpha - \sigma)\lambda_n\} = 0 = 1 \qquad \text{as } \sigma \to \alpha$

Hence $\limsup_{\sigma \to \alpha} \frac{\log \log M(\sigma, f^{(1)})}{\log \log(1 - e^{\sigma - \alpha})^{-1}} < P^*$

Again since $1 < \lambda^* < \infty$, them from (1.6) and (1.7) we have for any arbitrary small $\in > 0$,

$\log m(\sigma) > [\log(1 - e^{\sigma - \alpha})^{-1}]\lambda^* - \in$

And $\log[\lambda_{N(\sigma)}(1 - e^{\sigma - \alpha})] < \log\{\log(1 - e^{\sigma - \alpha})\}P^* + \in$

Therefore

$$\limsup_{\sigma \to \alpha} \frac{\log \lambda_{N\sigma 1}}{\log M(\sigma)} = 0 \tag{2.7}$$

Similarly

$$\lim_{\sigma \to \alpha} \frac{\log \lambda_{N(\sigma 1 f(1))}}{\log m(\sigma 1 f(1))} = 0 \tag{2.8}$$

Also from (2.2)

$$m(\sigma_1 f^{(1)}) \leq m(\sigma)\, \lambda_{N(\sigma 1f(1))} \tag{2.9}$$

Or

$$\lim_{\sigma\to\alpha} inf \quad \frac{\log\log m(\sigma, f^{(1)})}{\log\log(1-e^{\sigma-\alpha})^{-1}} \leq \lambda^*, \tag{2.10}$$

Or

$$\lim_{\sigma\to\alpha} inf \quad \frac{\log\log M(\sigma, f^{(1)})}{\log\log(1-e^{\sigma-\alpha})^{-1}} \tag{2.11}$$

Combining the results (2.4), (2.6) and (2.11) we get (2.1). Now we will prove some results which connect m (σ, f) and m (σ, $f^{(1)}$).

3. Theorem 2: Let f(s) = $\sum_{n=1}^{\infty} a_n e^{s\lambda n}$ be analytic in a half plane σ < α, of logarithmic order P^* and lower logarithmic order λ^*, where $1 < \lambda^* < P^* < \infty$, than

$$\lim_{\sigma\to\alpha} Sup \quad \frac{\log\left[\left\{m\frac{(\sigma, f^{(1)})}{m(\sigma, f)}\right\}(1-e^{\sigma-\alpha})\right]}{\log\log(1-e^{\sigma-\alpha})^{-1}} = P^* \tag{3.1}$$

Prof we have from (2.10)

$$m(\sigma_1 f^{(1)}) < m(\sigma)\, \lambda_{N(\sigma 1f(1))} \tag{3.2}$$

Again $m(\sigma) = |a_{N(\sigma)}| \exp(\sigma\, \lambda_{N(\sigma)}) = \frac{|a_{N(\sigma)}|\lambda_{N(\sigma)} \exp\{\sigma\, \lambda_{N(\sigma)}\}}{\lambda_{N(\sigma)}}$

Therefore,

$$m(\sigma) \leq m(\sigma_1 f^{(1)} / \lambda_{N(\sigma)} \tag{3.3}$$

From (3.2) and (3.3) we get

$$\lambda_{N(\sigma)} \leq \frac{m(\sigma_1 f^{(1)})}{m(\sigma_1 f)} \leq \lambda_{N(\sigma_1 f^{(1)})} \tag{3.4}$$

From the above inequality we have

$$\frac{\log[\lambda_{N(\sigma)}(1-e^{\sigma-\alpha})]}{\log\log(1-e^{\sigma-\alpha})^{-1}} < \frac{\log\left\{m(\sigma_1 f^{(1)})(1-\frac{e^{\sigma-\alpha)}}{m(\sigma_1 f)}\right\}}{\log\log(1-e^{\sigma-\alpha})^{-1}}$$

Taking superior limit and using (1.7) we get

$$P^* \leq \lim_{\sigma\to\alpha} sup \quad \frac{\log\left\{\left[m\frac{(\sigma_1 f^{(1)})}{m(\sigma_1 f)}\right](1-e^{\sigma-\alpha})\right\}}{\log\log(1-e^{\sigma-\alpha})^{-1}} \tag{3.5}$$

But the theorem 1, $f^{(1)}(s)$ is also analytic function of some logarithmic order P^* and lower order λ^*. Therefore, using other half of (3.4) we get

$$\lim_{\sigma \to \alpha} Sup \quad \frac{\log m \frac{(\sigma_1 f^{(1)})(1-e^{\sigma-\alpha})}{m(\sigma_1 f)}}{\log \log (1-e^{\sigma-\alpha})^{-1}} \qquad (3.6)$$

Combing (3.5) and 3.6), we get (3.1)

Corollary 1: If f(s) is of logarithmic order P* and lower logarithmic order λ^*, when $1 < \lambda^* < P^* < \infty$ and $f^{(n)}(s)$, then

$$\lim_{\sigma \to \alpha} Sup \quad \frac{\log \left[\frac{m(\sigma_1 f^{(n)})}{m(\sigma_1 f)}\right]^{\frac{1}{n}} (1-e^{\sigma-\alpha})}{\log \log (1 \leq -e^{\sigma-\alpha})^{-1}} = P^* \qquad (3.7)$$

Proof: Writing (3.4) for K^{th} derivative of f(s), we have

$$\lambda_{N(\sigma 1f(1))} \leq \frac{m(\sigma_1 f^{(k)}}{m(\sigma_1 f^{(k-1)})} \leq \lambda_{N(\sigma, f^k)} \qquad (3.8)$$

If follows from the definition that

$\lambda_{N(\sigma)} \leq \lambda_{N(\sigma 1f}(1)_{)} \leq ..... \lambda_{N(\sigma 1f}(n)_{)} \leq .....$

Putting K = 1, 2, ... n in (3.8) and multiplying all the inequalities thus obtained. We get

$$\lambda N(\sigma) \leq \left[\frac{m(\sigma_1 f^{(n)})}{m(\sigma_1 f)}\right]^{1/n} \leq \lambda_{N(\sigma 1f}(1)_{)} \qquad (3.9)$$

Now multiplying (3.9) by $(1 - e^{\sigma - \alpha})$ and taking limits we get the result (3.7) using (1.7) and (3.9).

Corollary 2: For $1 < P^* < \infty$ and for all $\sigma_1 \sigma_0 < \sigma < \alpha$ we have

$m(\sigma_1 f^{(n)} < m(\sigma) [\log (1 - e^{\sigma - \alpha})^{-1}]^{n P^* - \in} \{(1 - e^{\sigma - \alpha})^{-1}\}^n$

We obtain this result directly from (3.7)

4. Let $w(\sigma_1 \sigma) = \lambda_{N(\sigma 1f^{(N)})} - \lambda_{N(\sigma)}$

Then $w(\sigma_1 n)$ is a non negative, non decreasing difference function of σ for n = 1, 2,..... Let us assume that f(s) is function of logarithmic order P^* and lower logarithmic order λ^*, $1 < \lambda^* < P^* < \infty$ and

$w(\sigma 1\ n) \to \infty$ as $\sigma \to \alpha$ for any n = 1, 2, ...

Now we prime the following theorem

Theorem 3: For n = 1,2, ... and for some suitable value σ_0, we have

$$P^* = \frac{1}{n \log \log (1-e^{\sigma-\alpha})^{-1}} \int_{\sigma_0}^{\sigma} (w(y,n) + n2(y)\}dy$$

Where $\upsilon(y) = e^{y - \alpha} / 1 - e^{y - \alpha}$

Proof: For an analytic function f(s) defined by (1.1) we have (3.7)

$$\log m(\sigma) = 0(1) + \int_{\sigma_0}^{\sigma} \lambda_{N(y)}\,dy,\ \sigma_0 < \sigma < \alpha \quad (4.2)$$

Hence we have

$$\log m(\sigma_1\ f^{(n)}) = 0(1)\ D \int_{\sigma_0}^{\sigma} \lambda_{N(y, f^{(n)})}\,dy \quad (4.3)$$

Thus from (4.2) and (4.3) we have for all σ_1, $\sigma_0 < \sigma < \alpha$

$$\log \left\{\frac{m(\sigma_1 f^{(n)})}{m(\sigma_1 f)}\right\}^{1/n} = 0(1) + \frac{1}{n}\int_{\sigma_0}^{\sigma} w(y,n)\,dy$$

$$\text{or } \log \left\{ \left\{\frac{m(\sigma_1 f^{n})}{m(\sigma_1 f)}\right\}^{\frac{1}{n}} (1 - e^{\sigma-\alpha}) \right\}$$

$$= 0(1) + \frac{1}{n}\int_{\sigma_0}^{\sigma} w(y,n) + \int_{\sigma_0}^{\sigma} \frac{e^{y-\alpha}}{1 - e^{y-\alpha}}\,dy$$

$$= 0(1) + \frac{1}{n}\int_{\sigma_0}^{\sigma} [w(y,n) + nv(y)]\,dy.$$

Dividing both the sides by log log $(1 - e^{\sigma - \alpha})^{-1}$ and taking superior limit we get (4.1).

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