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ON SOME NEW TENSORS AND THEIR PROPERTIES IN A FOUR-DIMENSIONAL FINSLER SPACE-II

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ABSTRACT

Certain new tensors have been defined and studied in a Finsler space by Rastogi[4], while recently Rastogi[6] has introduced some new tensors D_{ijk} and Q_{ijk}etc. in a Finsler space of three dimensions in the following form:

$$D_{ijk} = D_{(1)}m_im_jm_k + D_{(2)}n_in_jn_k + \Sigma_{(ijk)}\{D_{(3)}\, m_im_jn_k\text{-}D_{(1)}\, m_in_jn_k\} \tag{1.1}$$

and

$$Q_{ijk} = \{D_{(1)\,/\,0} \text{-}3\, D_{(3)}\, h_0\}\, m_im_jm_k + (D_{(2)\,/\,0}\text{-}3D_{(1)}h_0)\, n_in_jn_k$$

$$+ \Sigma_{(ijk)}[\{(D_{\,(3)\,/\,0} +3\, D_{(1)}h_0\}\, m_im_jn_k \text{-} \{D_{(1)/\,0}+ (D_{(2)}\text{-}\ 2\, D_{(3)})\, h_0\}\, m_in_jn_k] \tag{1.2}$$

The tensor D_{ijk} so introduced satisfies $D_{ijk}\ l^i = 0$ and $D_{ijk}g^{jk} = D_i = D\ n_i$ and is similar to C_{ijk} while $Q_{ijk} = D_{ijk/\,0}$is similar to P_{ijk}. The purpose of the present paper is to introduce tensors ${}^1D_{ijk}$ and ${}^2D_{ijk}$in a Finsler spaceof four dimensions F^4 and study some of their properties. It is important to notice that in F^4 instead of one D_{ijk},we have two such tensors.

KEYWORDS: *New Tensors in Four-Dimensional Finsler Spaces*

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2. INTRODUCTION

Let F^4 be a Finsler space of four dimensions with the metric function L(x,y), and Moor's frame (l^i, m^i, n^i, p^i) such that the metric tensor $g_{ij}(x,y)$ and angular metric tensor are given as

$$g_{ij}(x,y) = l_il_j + m_im_j + n_in_j + p_ip_j,\ h_{ij} = m_im_j + n_in_j + p_ip_j \tag{2.1}$$

The h(hv)-torsion tensorC_{ijk}in F^4 is given as Rastogi [5]:

$$C_{ijk} = C_{(1)}\, m_im_jm_k + C_{(2)}n_in_jn_k + C_{(3)}\, p_ip_j\, p_k + C_{(4)}\textstyle\sum_{(ijk)} \{m_in_jn_k\}$$

$$+ C_{(5)} \textstyle\sum_{(ijk)} \{m_ip_j\, p_k\}+ C_{(6)}\sum_{(ijk)} \{ n_in_j\, p_k\} + C_{(7)}\sum_{(ijk)} \{n_ip_j\, p_k\} – (C_{(2)} + C_{(7)})$$

$$\textstyle\sum_{(ijk)} \{m_im_jn_k\} \text{-}(C_{(3)} + C_{(6)}) \sum_{(ijk)} \{m_im_j\, p_k\} + C_{(8)} \sum_{(ijk)} \{m_i\, (n_j\, p_k + n_kp_j)\}, \tag{2.2}$$

Where $C_{(1)}$ to $C_{(8)}$ are eight arbitrary scalars which can be determined.

The h-covariant derivative of vectors m_i, n_i and p_i are respectively given by

$$m_{i/r} = \alpha_rn_i + \beta_r\, p_i,\ n_{i/r} = \text{-}\alpha_r\, m_i + \gamma_r\, p_i,\ p_{i/r} = \text{-}\ \beta_r\, m_i – \gamma_rn_i, \tag{2.3}$$

Where vectors α_r, β_r and γ_r are unknown to be determined and are called three h-connection vectors in F^4.

The torsion tensor P_{ijk} in F^4 is expressed as

$$P_{ijk} = P_{(1)}\, m_i m_j m_k + P_{(2)} n_i n_j n_k + P_{(3)} p_i p_j\, p_k + P_{(4)} \sum_{(ijk)}\{m_i n_j n_k\}$$

$$+ P_{(5)} \sum_{(ijk)}\{m_i p_j\, p_k\} + P_{(6)} \sum_{(ijk)}\{n_i n_j\, p_k\} + P_{(7)} \sum_{(ijk)}\{n_i p_j\, p_k\}$$

$$+ P_{(8)} \sum_{(ijk)}\{m_i (\, p_j n_k + p_k n_j) + \{-(C_{(2)} + C_{(7)})_{/\,0} + (C_{(1)} - 2C_{(4)})\, \alpha_0$$

$$- 2C_{(8)}\, \beta_0 + (C_{(3)} + C_{(6)})\, \gamma_0\} \sum_{(ijk)}\{m_i m_j n_k\} + \{-(C_{(3)} + C_{(6)})_{/\,0} - 2\, C_{(8)}\, \alpha_0$$

$$+ (C_{(1)} - 2C_{(5)})\, \beta_0 - (C_{(2)} + C_{(7)})\, \gamma_0\} \sum_{(ijk)}\{m_i m_j\, p_k\} \qquad (2.4)$$

Where

$$P_{(1)} = C_{(1)\,/\,0} - 3\, C_{(7)}\, \alpha_0 + 3(C_{(3)} + C_{(6)})\, \beta_0,$$

$$P_{(2)} = C_{(2)\,/\,0} + 3\, C_{(4)}\, \alpha_0 - 3\, C_{(6)}\, \gamma_0,$$

$$P_{(3)} = C_{(3)\,/\,0} + 3\, C_{(5)}\, \beta_0 + 3\, C_{(7)}\, \gamma_0,$$

$$P_{(4)} = C_{(4)\,/\,0} - (3C_{(2)} + 2\, C_{(7)})\, \alpha_0 - C_{(6)}\, \beta_0 - 2\, C_{(8)}\, \gamma_0$$

$$P_{(5)} = C_{(5)\,/\,0} - C_{(7)}\, \alpha_0 - (3C_{(3)} - 2C_{(6)})\, \beta_0 + 2\, C_{(8)}\, \gamma_0\},$$

$$P_{(6)} = C_{(6)\,/\,0} + 2\, C_{(8)}\, \alpha_0 + C_{(4)}\, \beta_0 + (C_{(2)} - 2\, C_{(7)})\, \gamma_0,$$

$$P_{(7)} = C_{(7)\,/\,0} + C_{(5)}\, \alpha_0 + C_{(8)}\, \beta_0 + (2C_{(6)} - C_{(3)})\, \gamma_0,$$

$$P_{(8)} = C_{(8)\,/\,0} - (C_{(3)} + 2C_{(6)})\, \alpha_0 - (C_{(2)} + 2C_{(7)})\, \beta_0 + (C_{(4)} - C_{(5)})\, \gamma_0.$$

The v-covariant derivative of vectors l_i, m_i, n_i and p_i are respectively given by

$$L\, l_{i//j} = h_{ij},\quad L\, m_{i//j} = -\, l_i m_j + n_i u_j + p_i v_j,$$

$$L\, n_{i//j} = -\, l_i n_j - m_i u_j + p_i w_j,\quad L\, p_{i//j} = -(l_i p_j + m_i v_j + n_i w_j) \qquad (2.5)$$

Where

$$L\, e_{\alpha)\, i/j} = V_{\alpha)\, \beta\gamma}\, e_{\beta)\, i} e_{\gamma)\, j} \text{ and } u_i = u\, e_{\gamma)\, i},\; v_i = v\, e_{\gamma)\, i},\; w_i = w\, e_{\gamma)\, i}.$$

3. SOME NEW TENSORS OF SECOND ORDER AND THEIR h-COVARIANT DERIVATIVES

Definition 3.1: In a Finsler space of four dimensions F^4, we define non-zero second order symmetric tensors ${}^1A_{ij}(x,y)$, ${}^2A_{ij}(x,y)$, ${}^3A_{ij}(x,y)$, ${}^1B_{ij}(x,y)$, ${}^2B_{ij}(x,y)$ and ${}^3B_{ij}(x,y)$, given by

$${}^1A_{ij}(x,y) = \sum_{(ij)}\{l_i m_j\},\; {}^2A_{ij}(x,y) = \sum_{(ij)}\{l_i n_j\},\; {}^3A_{ij}(x,y) = \sum_{(ij)}\{l_i p_j\} \qquad (3.1)$$

and

$${}^1B_{ij}(x,y) = \sum_{(ij)}\{n_i m_j\},\; {}^2B_{ij}(x,y) = \sum_{(ij)}\{p_i m_j\},\; {}^3B_{ij}(x,y) = \sum_{(ij)}\{n_i p_j\} \qquad (3.2)$$

From equations (3.1) and (3.2), their h-covariant derivatives give

$${}^1A_{ij/k} = \alpha_k {}^2A_{ij} + \beta_k {}^3A_{ij},\; {}^2A_{ij/k} = -\, \alpha_k {}^1A_{ij} + \gamma_k {}^3A_{ij},\; {}^3A_{ij/k} = -\, \beta_k {}^1A_{ij} - \gamma_k {}^2A_{ij} \qquad (3.3)$$

and

$^1B_{i/|k} = -2\ \alpha_k(m_i m_j - n_i n_j) + \beta_k{}^3B_{ij} + \gamma_k{}^2B_{ij},$

$^2B_{ij/k} = \alpha_k{}^3B_{ij} - 2\ \beta_k\ (m_i m_j - p_i p_j) - \gamma_k{}^1B_{ij},$

$^3B_{ij/k} = -\ \alpha_k{}^2B_{ij} - \beta_k{}^1B_{ij} - 2\ \gamma_k(n_I n_j - p_i p_j)$ (3.4)

From equations (3.3) and (3.4) we can obtain

Theorem 3.1: In a four dimensional Finsler space F^4, tensors $^1A_{ij/k}$, $^2A_{ij/k}$, $^3A_{ij/k}$, $^1B_{ij/k}$, $^2B_{ij/k}$ and $^3B_{ij/k}$ satisfy equations

$$^1A_{ij/k} + {}^2A_{ij/k} + {}^3A_{ij/k} = -\ (\alpha_k + \beta_k)\ {}^1A_{ij} - (\gamma_k - \alpha_k)\ {}^2A_{ij} + (\beta_k + \gamma_k)\ {}^3A_{ij} \quad (3.5)$$

and

$$^1B_{ij/k} + {}^2B_{ij/k} + {}^3B_{ij/k} = -\ (\beta_k + \gamma_k)({}^1B_{ij} - 2p_i p_j) + (\gamma_k - \alpha_k)({}^2B_{ij} - 2\ n_i n_j)$$

$$+(\alpha_k + \beta_k)({}^3B_{ij} - 2m_i m_j) \quad (3.6)$$

Definition 3.2: In a Finsler space of four dimensions F^4, we define non-zero second order symmetric tensors $^1U_{ij}(x,y)$, $^2U_{ij}(x,y)$, $^3U_{ij}(x,y)$, $^1T_{ij}(x,y)$, $^2T_{ij}(x,y)$ and $^3T_{ij}(x,y)$, given by

$$^1U_{ij} = m_i\ m_j - n_i n_j,\ {}^2U_{ij} = n_i n_j - p_i p_j,\ {}^3U_{ij} = p_i p_j - m_i m_j \quad (3.7)$$

and

$$^1T_{ij} = m_i m_j + n_i n_j,\ {}^2T_{ij} = n_i n_j + p_i p_j,\ {}^3T_{ij} = p_i p_j + m_o m_j \quad (3.8)$$

The h-covariant derivatives of these tensors are given by

$^1U_{ij/k} = 2\ \alpha_k{}^1B_{ij} + \beta_k{}^2B_{ij} - \gamma_k{}^3B_{ij},\ {}^2U_{ij/k} = -\ \alpha_k{}^1B_{ij} + \beta_k{}^2B_{ij} + 2\ \gamma_k{}^3B_{ij},$

$^3U_{ij/k} = -\ \alpha_k{}^1B_{ij} - 2\ \beta_k{}^2B_{ij} - \gamma_k{}^3B_{ij},$ (3.9)

$$^1T_{ij/k} = \beta_k{}^2B_{ij} + \gamma_k{}^3B_{ij},\ {}^2T_{ij/k} = -\ \alpha_k{}^1B_{ij} - \beta_k{}^2B_{ij},\ {}^3T_{ij/k} = \alpha_k{}^1B_{ij} - \gamma_k{}^3B_{ij}. \quad (3.10)$$

Definition 3.3: In a Finsler space of four dimensions F^4, we define non-zero second order skew-symmetric tensors $^1E_{ij}(x,y)$, $^2E_{ij}(x,y)$, $^3E_{ij}(x,y)$, $^1F_{ij}(x,y)$, $^2F_{ij}(x,y)$ and $^3F_{ij}(x,y)$, given by

$$^1E_{ij} = l_i m_j - m_i l_j,\ {}^2E_{ij} = l_i n_j - n_i l_j,\ {}^3E_{ij} = l_i p_j - p_i l_j, \quad (3.11)$$

$$^1F_{ij} = m_i n_j - m_j n_i,\ {}^2F_{ij} = m_i p_j - m_j\ p_i,\ {}^3F_{ij} = n_i p_j - n_j\ p_i \quad (3.12)$$

The h-covariant derivatives of these tensors satisfy

$$^1E_{ij/k} = \alpha_k{}^2E_{ij} + \beta_k{}^3E_{ij},\ {}^2E_{ij/k} = -\ \alpha_k{}^1E_{ij} + \gamma_k{}^3E_{ij},\ {}^3E_{ij/k} = -\ \beta_k{}^1E_{ij} + \gamma_k{}^2E_{ij} \quad (3.13)$$

and

$$^1F_{ij/k} = \gamma_k{}^2E_{ij} - \beta_k{}^3F_{ij},\ {}^2F_{ij/k} = \alpha_k{}^3F_{ij} - \gamma_k{}^1F_{ij},\ {}^3F_{ij/k} = \beta_k{}^1F_{ij} - \alpha_k{}^2F_{ij} \quad (3.14)$$

From equations (3.13) and (3.14) we can obtain

Theorem 3.2: In a four dimensional Finsler space F^4, tensors $^1E_{ij/k}$, $^2E_{ij/k}$, $^3E_{ij/k}$, $^1F_{ij/k}$, $^2F_{ij/k}$ and $^3F_{ij/k}$ satisfy equations

$$^1E_{ij/k} + {}^2E_{ij/k} + {}^3E_{ij/k} = \alpha_k({}^2E_{ij} - {}^1E_{ij}) + \beta_k({}^3E_{ij} - {}^1E_{ij}) + \gamma_k({}^3E_{ij} - {}^2E_{ij}) \quad (3.15)$$

and

$$ {}^1F_{ij/k} + {}^2F_{ij/k} + {}^3F_{ij/k} = \alpha_k({}^3F_{ij} – {}^2F_{ij}) + \beta_k({}^1F_{ij} – {}^3F_{ij}) + \gamma_k({}^2F_{ij} – {}^1F_{ij}) \quad (3.16)$$

4. V-COVARIANT DERIVATIVES OF SECOND ORDER TENSORS

The v-covariant derivatives of second order tensors defined in (3.1) and (3.2) give

$$ {}^1A_{ij//k} = L^{-1}\{ 2(m_i m_j - l_i l_j)\, m_k + n_k{}^1B_{ij} + p_k{}^2B_{ij} + u_k{}^2A_{ij} + v_k{}^3A_{ij}\},$$

$$ {}^2A_{ij//k} = L^{-1}\{ 2\,(n_i n_j – l_i l_j)\, n_k + m_k{}^1B_{ij} + p_k{}^3B_{ij} – u_k{}^1A_{ij} + w_k{}^3A_{ij}\},$$

$$ {}^3A_{ij//k} = L^{-1}\{ 2(p_i p_j – l_i l_j)\, p_k + m_k{}^2B_{ij} + n_k{}^3B_{ij} - v_k{}^1A_{ij} – w_k{}^2A_{ij} \}, \quad (4.1)$$

and

$$ {}^1B_{ij//k} = L^{-1}\{2(n_i n_j – m_i m_j)u_k + v_k{}^3B_{ij} – w_k{}^2B_{ij} – m_k{}^2A_{ij} – n_k{}^1A_{ij}\},$$

$$ {}^2B_{ij//k} = L^{-1}\{2(p_i p_j - m_i m_j)v_k + u_k{}^3B_{ij} – w_k{}^1B_{ij} – m_k{}^3A_{ij} – p_k{}^1A_{ij}\},$$

$$ {}^3B_{ij//k} = L^{-1}\{2(p_i p_j – n_i n_j)\, w_k – u_k{}^2B_{ij} – v_k{}^1B_i – n_k{}^3A_{ij} – p_k{}^2A_{ij}\} \quad (4.2)$$

From equations (4.1) and (4.2), we can obtain

Theorem 4.1: In a four dimensional Finsler space F^4, tensors ${}^1A_{ij//k}$, ${}^2A_{ij//k}$, ${}^3A_{ij//k}$, ${}^1B_{ij//k}$, ${}^2B_{ij//k}$ and ${}^3B_{ij//k}$ satisfy following equations

$$ {}^1A_{ij//k} + {}^2A_{ij//k}, + {}^3A_{ij//k} = L^{-1}[2\{(m_i m_j – l_i l_j)m_k + (n_i n_j - l_i l_j)n_k + (p_i p_j - l_i l_j)p_k\}$$

$$ + m_k({}^1B_{ij} + {}^2B_{ij}) + n_k({}^3B_{ij} + {}^1B_{ij}) + p_k({}^2B_{ij} + {}^3B_{ij})$$

$$ + u_k\,({}^2A_{ij} – {}^1A_{ij}) + v_k({}^3A_{ij} – {}^1A_{ij}) + w_k\,({}^3A_{ij} – {}^2A_{ij})] \quad (4.3)$$

and

$$ {}^1B_{ij//k} + {}^2B_{ij//k}, + {}^3B_{ij//k} = L^{-1}[2\{(n_i n_j - m_i m_j)u_k + (p_i p_j - m_i m_j)v_k$$

$$ + (p_i p_j - n_i n_j)w_k\} – m_k({}^2A_{ij} + {}^3A_{ij}) - n_k({}^3A_{ij} + {}^1A_{ij})$$

$$ - p_k({}^1A_{ij} + {}^2A_{ij}) + u_k({}^3B_{ij} - {}^2B_{ij}) + v_k({}^3B_{ij} - {}^1B_{ij})$$

$$ - w_k({}^1B_{ij} + {}^2B_{ij})] \quad (4.4)$$

The v-covariant derivatives of second order tensors defined in (3.7) and (3.8) give

$$ {}^1U_{ij//k} = L^{-1}\{-m_k{}^1A_{ij} + n_k{}^2A_{ij} + u_k({}^1B_{ij} + 2n_i n_j) + v_k{}^2B_{ij} + w_k{}^3B_{ij}\},$$

$$ {}^2U_{ij//k} = L^{-1}\{- n_k{}^2A_{ij} + p_k{}^3A_{ij} - 2\, u_k n_i n_j + v_k{}^2B_{ij}\},$$

$$ {}^3U_{ij//k} = L^{-1}\{m_k{}^1A_{ij} – p_k{}^3A_{ij} – u_k{}^1B_{ij} – 2\, v_k{}^2B_{ij} – w_k{}^3B_{ij}\},$$

$$ {}^1T_{ij//k} = L^{-1}\{-m_k{}^1A_{ij} - n_k{}^2A_{ij} + u_k(\, {}^1B_{ij} – 2\, n_i n_j) + v_k{}^2B_{ij} - w_k{}^3B_{ij}\},$$

$$ {}^2T_{ij//k} = L^{-1}\{- n_k{}^2A_{ij} - p_k{}^3A_{ij} - 2\, u_k n_i n_j - v_k{}^2B_{ij} – 2\, w_k{}^3B_{ij}\},$$

$$ {}^3T_{ij//k} = L^{-1}\{- m_k{}^1A_{ij} – p_k{}^3A_{ij} + u_k{}^1B_{ij} – w_k{}^3B_{ij}\} \quad (4.5)$$

While the v-covariant derivatives of second order tensors defined in (3.11) and (3.12) give

$$ {}^1E_{ij//k} = L^{-1}(-n_k{}^1F_{ij} + p_k{}^2F_{ij} + u_k{}^2E_{ij} + v_k{}^3E_{ij}),$$

$$^2E_{ij//k} = L^{-1}(m_k {}^1F_{ij} - p_k {}^3F_{ij} - u_k {}^1E_{ij} + w_k {}^3E_{ij}),$$

$$^3E_{ij//k} = L^{-1}(m_k {}^2F_{ij} + n_k {}^3F_{ij} - v_k {}^1E_{ij} - w_k {}^2E_{ij}),$$

$$^1F_{ij//k} = L^{-1}(-m_k {}^2E_{ij} + n_k {}^1E_{ij} - v_k {}^3F_{ij} + w_k {}^2F_{ij}),$$

$$_{2Fij//k} = L^{-1}(- m_k {}^3E_{ij} + p_k {}^1E_{ij} + u_k {}^3F_{ij} - w_k {}^1F_{ij}),$$

$$^3F_{ij//k} = L^{-1}(- n_k {}^3E_{ij} + p_k {}^2E_{ij} - u_k {}^2F_{ij} + v_k {}^1F_{ij}) \quad .(4.6)$$

5. D-TENSOR OF FIRST KIND

In F^4, there exists D-tensors of two kind. In this section we shall define D-tensor of first kind. Let $^1D_{ijk}$, be the D-tensor of first kind, which is such that $^1D_{ijk}\, l^i = 0$ and $^1D_{ijk}g^{jk} = {}^1D_i = {}^1D\, n_i$. Any third order tensor in F^4, satisfying above properties can be expressed as

$$\begin{aligned}^1D_{ijk} = {} & D_{(1)}\, m_im_jm_k + D_{(2)}n_in_jn_k + D_{(3)}\, p_ip_j\, p_k + D_{(4)} \textstyle\sum_{(ijk)} \{m_im_jn_k\} \\ & + D_{(5)}\textstyle\sum_{(ijk)}\{m_im_j\, p_k\} + D_{(6)}\textstyle\sum_{(ijk)}\{n_in_jm_k\} + D_{(7)} \textstyle\sum_{(ijk)}\{m_i\, (n_j\, p_k + n_kp_j)\} \\ & + D_{8)}\textstyle\sum_{(ijk)}\{n_in_j\, p_k\} + D_{(9)}\textstyle\sum_{(ijk)}\{p_ip_jm_k\} + D_{(10)}\textstyle\sum_{(ijk)}\{p_ip_jn_k\} \end{aligned} \quad (5.1)$$

Multiplying equation (5.1) by g^{jk}, we obtain on simplification

$$^1D_i = (D_{(1)} + D_{(6)} + D_{(9)})m_i + (D_{(2)} + D_{(4)} + D_{(10)})n_i + (D_{(3)} + D_{(5)} + D_{(8)})p_i \quad (5.2)$$

Which implies

$$D_{(1)} + D_{(6)} + D_{(9)} = 0,\ D_{(2)} + D_{(4)} + D_{(10)} = {}^1D,\ D_{(3)} + D_{(5)} + D_{(8)} = 0 \quad (5.3)$$

Substituting from (5.3) in (5.1), we can write

$$\begin{aligned}^1D_{ijk} = {} & D_{(1)}\, m_im_jm_k + D_{(2)}n_in_jn_k + D_{(3)}\, p_ip_j\, p_k + D_{(4)} \textstyle\sum_{(ijk)} \{m_im_jn_k\} \\ & + D_{(5)}\textstyle\sum_{(ijk)}\{m_im_j\, p_k\} + D_{(6)}\textstyle\sum_{(ijk)}\{n_in_jm_k\} + D_{(7)} \textstyle\sum_{(ijk)}\{m_i\, (n_j\, p_k + n_kp_j)\} \\ & -(D_{(3)} + D_{(5)})\textstyle\sum_{(ijk)}\{n_in_j\, p_k\} - (D_{(1)} + D_{(6)})\textstyle\sum_{(ijk)}\{p_ip_jm_k\} \\ & + ({}^1D\text{-}D_{(2)} - D_{(4)})\textstyle\sum_{(ijk)}\{p_ip_jn_k\} \end{aligned} \quad (5.4)$$

From equation (5.4), we can give

Definition 5.1: In a four- dimensional Finsler space F^4, the tensor $^1D_{ijk}$ defined by equation (5.4) is called D-tensor of first kind.

This tensor can also be expressed as

$$^1D_{ijk} = \textstyle\sum_{(ijk)}\{m_iX_{jk} + n_iY_{jk} + p_iZ_{jk}\}, \quad (5.5)$$

Where

$$\begin{aligned}X_{jk} = {} & (1/3)\, D_{(1)}m_jm_k + (1/2)\, D_{(4)}\, (m_jn_k + m_kn_j) \\ & + (1/2)\, D_{(5)}(m_j\, p_k + m_kp_j) + (1/3)\, D_{(7)}(p_jn_k + p_kn_j) \end{aligned} \quad (5.6)$$

$$Y_{jk} = (1/3)\, D_{(2)}n_jn_k - (1/2)(D_{(3)} + D_{(5)})(n_j\, p_k + n_kp_j)$$

$$+(1/2)\, D_{(6)}\, (m_j n_k + m_k n_j) + (1/3)\, D_{(7)}(m_j\, p_k + m_k p_j) \tag{5.7}$$

$$Z_{jk} = (1/3)\, D_{(3)} p_j\, p_k + (1/2)({}^1D - D_{(2)} - D_{(4)})(n_j\, p_k + n_k p_j)$$

$$- (1/2)(D_{(1)} + D_{(6)})(m_j\, p_k + m_k p_j) + (1/3)\, D_{(7)}(m_j n_k + m_k n_j) \tag{5.8}$$

are symmetric tensors of second order.

These tensors defined above satisfy

$$X_{jk} m^k = (1/3)\, D_{(1)} m_j + (1/2)\, D_{(4)} n_j + (1/2)\, D_{(5)} p_j,$$

$$X_{jk} n^k = (1/2)\, D_{(4)} m_j + (1/3)\, D_{(7)} p_j,$$

$$X_{jk}\, p^k = (1/2)\, D_{(5)} m_j + (1/3)\, D_{(7)} p_j,$$

$$Y_{jk} m^k = (1/2)\, D_{(6)} n_j + (1/3)\, D_{(7)} p_j,$$

$$Y_{jk} n^k = (1/3) D_{(2)} n_j - (1/2)(D_{(3)} + D_{(5)}) p_j + (1/2)\, D_{(6)} m_j,$$

$$Y_{jk}\, p^k = -(1/2)\, (D_{(3)} + D_{(5)}) n_j + (1/3)\, D_{(7)} m_j,$$

$$Z_{jk} m^k = -(1/2)(D_{(1)} + D_{(6)})\, p_j + (1/3)\, D_{(7)} n_j,$$

$$Z_{jk} n^k = (1/2)({}^1D - D_{(2)} - D_{(4)}) p_j + (1/3)\, D_{(7)}\, m_j,$$

$$Z_{jk}\, p^k = (1/3)\, D_{(3)} p_j + (1/2)({}^1D - D_{(2)} - D_{(4)})\, n_j - (1/2)(D_{(1)} + D_{(6)})\, m_j. \tag{5.9}$$

6. D-TENSOR OF SECOND KIND

In this section we shall define asymmetric D-tensor of second kindand denote it by ${}^2D_{ijk}$, which satisfies ${}^2D_{ijk}\, l^i = 0$ and ${}^2D_{ijk} g^{jk} = {}^2D\, p_i$. Any third order tensor satisfying these properties can be expressed as

$$\begin{aligned} {}^2D_{ijk} = {} & D^*_{(1)}\, m_i m_j m_k + D^*_{(2)} n_i n_j n_k + D^*_{(3)}\, p_i p_j\, p_k + D^*_{(4)} \textstyle\sum_{(ijk)} \{m_i m_j n_k\} \\ & + D^*_{(5)} \textstyle\sum_{(ijk)} \{m_i n_j n_k\} + D^*_{(6)} \textstyle\sum_{(ijk)} \{m_i m_j\, p_k\} + D^*_{(7)} \textstyle\sum_{(ijk)} \{m_i\, (n_j\, p_k + n_k p_j)\} \\ & + D^*_{(8)} \textstyle\sum_{(ijk)} \{m_i p_j\, p_k\} + D^*_{(9)} \textstyle\sum_{(ijk)} \{n_i p_j\, p_k\} + D^*_{(10)} \textstyle\sum_{(ijk)} \{n_i n_j\, p_k\} \end{aligned} \tag{6.1}$$

Multiplying equation (6.1) by g^{jk}, we obtain on simplification

$$\begin{aligned} {}^2D_i = {} & (D^*_{(1)} + D^*_{(5)} + D^*_{(8)})\, m_i + (D^*_{(2)} + D^*_{(4)} + D^*_{(9)}) n_i \\ & + (D^*_{(3)} + D^*_{(6)} + D^*_{(10)}) p_i \end{aligned} \tag{6.2}$$

Which gives

$$D^*_{(1)} + D^*_{(5)} + D^*_{(8)} = 0,\; D^*_{(2)} + D^*_{(4)} + D^*_{(9)} = 0,$$

$$D^*_{(3)} + D^*_{(6)} + D^*_{(10)} = {}^2D \tag{6.3}$$

Substituting from equation (6.3) in (6.1), we get

$$\begin{aligned} {}^2D_{ijk} = {} & D^*_{(1)}\, m_i m_j m_k + D^*_{(2)} n_i n_j n_k + D^*_{(3)}\, p_i p_j\, p_k + D^*_{(4)} \textstyle\sum_{(ijk)} \{m_i m_j n_k\} \\ & + D^*_{(5)} \textstyle\sum_{(ijk)} \{m_i n_j n_k\} + D^*_{(6)} \textstyle\sum_{(ijk)} \{m_i m_j\, p_k\} + D^*_{(7)} \textstyle\sum_{(ijk)} \{m_i\, (n_j\, p_k + n_k p_j)\} \\ & -(D^*_{(1)} + D^*_{(5)}) \textstyle\sum_{(ijk)} \{m_i p_j\, p_k\} + (D^*_{(2)} + D^*_{(4)}) \textstyle\sum_{(ijk)} \{n_i p_j\, p_k\} \end{aligned}$$

$$+ ({}^2D - D^*_{(3)} - D^*_{(6)}) \sum_{(ijk)}\{n_i n_j p_k\} \quad (6.4)$$

From equation (6.4), we give

Definition 6.1: In a four- dimensional Finsler space F^4, the tensor ${}^2D_{ijk}$ defined by equation (6.4) is called D-tensor of second kind.

This tensor can also be expressed as

$${}^2D_{ijk} = \sum_{(ijk)} \{X^*_{jk} m_i + Y^*_{jk} n_i + Z^*_{jk} p_i\}, \quad (6.5)$$

Where

$$X^*_{jk} = (1/3)\, D^*_{(1)} m_j m_k + (1/2)\, D^*_{(4)}(m_j n_k + m_k n_j)$$

$$+ (1/2)\, D^*_{(6)}(m_j p_k + m_k p_j) + (1/3)\, D^*_{(7)}(n_j p_k + n_k p_j) \quad (6.6)$$

$$Y^*_{jk} = (1/3)\, D^*_{(2)} n_j n_k + (1/2)\, D^*_{(5)} (m_j n_k + m_k n_j)$$

$$+ (1/2)\, ({}^2D - D^*_{(3)} - D^*_{(6)})(n_j p_k + n_k p_j) + (1/3)\, D^*_{(7)}(m_j p_k + m_k p_j) \quad (6.7)$$

$$Z^*_{jk} = (1/3)\, D^*_{(3)} p_j p_k + (1/2)\, (D^*_{(2)} + D^*_{(4)})\, (n_j p_k + n_k p_j)$$

$$- (1/2)\, (D^*_{(1)} + D^*_{(5)})(m_j p_k + m_k p_j) + (1/3)\, D^*_{(7)}(m_j n_k + m_k n_j) \quad (6.8)$$

are symmetric tensorsof second order. These tensors satisfy

$$X^*_{jk} m^k = (1/3)\, D^*_{(1)} m_j + (1/2)\, D^*_{(4)} n_j + (1/2)\, D^*_{(6)} p_j,$$

$$X^*_{jk} n^k = (1/2)\, D^*_{(4)} m_j + (1/3) D^*_{(7)} p_j$$

$$X^*_{jk}\, p^k = (1/2)\, D^*_{(6)} m_j + (1/3)\, D^*_{(7)} p_j,$$

$$Y^*_{jk} m^k = (1/2)\, D^*_{(5)} n_j + (1/3)\, D^*_{(7)} p_j,$$

$$Y^*_{jk} n^k = (1/3)\, D^*_{(2)} n_j + (1/2)\, D^*_{(5)} m_j + (1/2)\, ({}^2D - D^*_{(3)} - D^*_{(6)})\, p_j,$$

$$Y^*_{jk}\, p^k = (1/2)\, ({}^2D - D^*_{(3)} - D^*_{(6)}) n_j + (1/3)\, D^*_{(7)} m_j,$$

$$Z^*_{jk} m^k = - (1/2)\, (D^*_{(1)} + D^*_{(5)}) p_j + (1/3)\, D^*_{(7)} n_j,$$

$$Z^*_{jk} n^k = (1/2)\, (D^*_{(2)} + D^*_{(4)})\, p_j + (1/3)\, D^*_{(7)} m_j,$$

$$Z^*_{jk}\, p^k = (1/3)\, D^*_{(3)} p_j + (1/2)\, (D^*_{(2)} + D^*_{(4)})\, n_j - (1/2)\, (D^*_{(1)} + D^*_{(5)})\, m_j. \quad (6.9)$$

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