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BAYESIAN PREDICTION BOUNDS OF DOUBLY TYPE-II CENSORED SAMPLES FOR A NEW BATHTUB SHAPE FAILURE RATE LIFE TIME MODEL

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ABSTRACT

This article considers the analysis of a doubly Type-II censored data, drawn from a new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Where, the one- and two-sample Bayesian prediction schemes have been used for driving prediction bounds of ordering unobserved lifetimes from the underlying distribution. For illustration purposes, some numerical examples are given. The accuracy of the resulting Bayesian prediction bounds, as well as percentage coverage, for future unobserved ordered lifetimes are investigated. This is done via extensive Monte Carlo simulation experiments based on 10,000 runs each.

KEYWORDS: Doubly Type-II Censored Sample, One-And Two-Sample Predictions, Bayesian Predic- Tion, Percentage Coverage, Monte Carlo Simulation

1. INTRODUCTION

In many statistical problems, researchers need to use the past results of data, which are related to some distribution, to predict future data from the same distribution. Such problems have received much attention by many authors, including Dunsmore (1974), Aitchison and Dunsmore (1975) and Geisser (1993). Bayesian prediction intervals for future observations have been discussed in several articles, see e. g. Others, including Jaheen (2002), AL-Hussaini and Ahmad (2003), Howlader et al. (2007), Ahmad (2011), Mohie El-Din and Shafay (2011), Balakrishnan and Shafay (2012), Ateya (2013), AL-Hussaini et al. (2015) Niazi (2016) and Niazi and Abd-Elrahman (2015). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function has been introduced by Chen (2000). We shall denote this distribution by $Chen(\lambda\,;\beta)$. Its cumulative distribution function (cdf) and the probability density function (pdf) are, respectively, given by

$$F_X(x) = 1 - e^{-\lambda\left(e^{x^\beta}-1\right)}, \tag{1}$$

and

$$f_X(x) = \lambda\,\beta\,x^{\beta-1}e^{x^\beta-\lambda\left(e^{x^\beta}-1\right)}, x > 0, \lambda > 0, \beta > 0. \tag{2}$$

Suppose that n items, $x_1, x_2, \cdots, x_n$, are put on a life test. The test starts at time 0, but only the ordered lifetimes $x_{(s)} < x_{(s+1)} < \cdots < x_{(r)}, 1 < s < r < n$, are observed. This means that, the first $(s-1)$ ordered lifetimes, as well as the last $(n-r)$ ordered lifetimes were not observed. The incomplete ordered observed lifetimes are referred to as doubly Type-II censored sample. As a special case, it is will known that, when s = 1, this sample would be the well known Type-II right censored sample.

On the other hand, when $s = 1$ and $n = r$ the resulting sample is then the complete sample.

In the Bayesian prediction bounds, we shall consider two of the most commonly used schemes, namely, the one- and two-sample schemes.

In the one-sample scheme, based on a doubly Type-II censored sample, each of $x_{(r+1)}, x_{(r+2)}, \cdots, x_{(n)}$ As unobserved lifetimes need to be predicted, see Geisser (1993).

In the two-sample scheme, based on a doubly Type-II censored sample from $Chen(\lambda, \beta)$ distribution (past sample), a new ordered sample of size m (future sample) from the same distribution needs to be predicted. In another word, let $x_{(s)} < x_{(s+1)} < \cdots < x_{(r)}$ be ordered observed lifetimes, out of n, we need to predict the boundaries of each member of a new ordered sample, see Dunsmore (1974).

In this article, Bayesian prediction intervals for future ordered lifetimes, having the $Chen(\lambda, \beta)$ distribution obtained, will be done based on a given doubly Type-II censored sample from this distribution. Assuming that the parameter β is known, Section 2 is devoted to obtaining Bayesian prediction bounds (BPBs) for future observations from the $Chen(\lambda, \beta)$ distribution. Section 3 is concerned with the BPBs problem, assuming that both of the two parameters λ and β are unknown. The one and two-sample schemes are considered in both of the one and the two parameters cases. A numerical example is given to illustrate the procedures, and the accuracy of prediction intervals is investigated via Monte Carlo simulation.

2. ONE PARAMETER CASE

Let $x_{(s)}, x_{(s+1)}, \cdots, x_{(r)}$ be a doubly Type-II censored sample from a $Chen(\lambda, \beta)$ distribution, whose cdf and pdf are as given by (1) and (2), respectively. The likelihood function for the parameters λ and β, is then given by

$$L(\lambda, \beta;\ \underline{x}) \propto (\lambda\beta)^{r-s+1} \sum_{k=0}^{s-1} \omega_s(k) \prod_{i=s}^{r} x_i^{\beta-1}\, exp\left\{\sum_{i=s}^{r} x_i^{\beta} - \lambda\, T(\underline{x})\right\}, x_s > \quad \textbf{(3)}$$

where $x_{(s)}, x_{(s+1)}, \cdots, x_{(r)}, \omega_s(k) = (-1)^k \binom{s-1}{k}$ and

$$T(\underline{x}) = \sum_{i=s}^{r} \left(e^{x_{(i)}^{\beta}} - 1\right) + k\left(e^{x_{(s)}^{\beta}} - 1\right) + (n-r)(e^{x_{(r)}^{\beta}} - 1)$$

Assuming that the parameter β is known, in view of (3), it might be clear that the parameter λ can have a $gamma(a_1, b_1)$ conjugate prior distribution with pdf of the form

$$g_1(\lambda) = \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda^{a_1-1} e^{-b_1\lambda}, \lambda > 0, (a_1, b_1 > 0). \quad \textbf{(4)}$$

By using (3) and (4) the posterior density function for λ can be written as follows

$$g_1^*(\lambda | \beta, \underline{x}) = C_1 \lambda^{r-s+a_1} \sum_{k=0}^{s-1} \omega_s(k)\, exp\{-\lambda(T(\underline{x}) + b_1)\} \quad \textbf{(5)}$$

where C_1 is the normalizing constant given by $(T(\underline{x}) + b_1)$

$$C_1^{-1} = \Gamma(d_0) \sum_{k=0}^{s-1} \omega_s(k)\ \left(T(\underline{x}) + b_1\right)^{-d_0}, \quad \textbf{(6)}$$

where $d_0 = r - s + a_1 + 1, \omega_s(k)$ and $T(\underline{x})$ is as given in (3).

As can be seen from (5), the posterior density is a gamma $(d_0, b_1 + T(\underline{x}))$ distribution.

In what follows, we shall obtain Bayesian prediction intervals for future observations in either the one- or two-sample schemes.

2.1. Bayesian One-Sample Prediction

Let $y_{(a)} \equiv x_{(r+a)}, 1 \leq a \leq n-r$ denotes the future (unobserved) lifetime of the a^{th} component to fail. The conditional probability density functions of y_a given the parameter β and that r components can be written as (see Arnold et al. (1992))

$$h_Y(y_{(a)};\lambda,\beta) = D_1(a)\left[F_X(y_{(a)};\lambda,\beta) - F_X(x_{(r)};\lambda,\beta)\right]^{a-1}\left[1 - F_X(y_{(a)};\lambda,\beta)\right]^{n-r-a} \times \left[1 - F_X(x_{(r)};\lambda,\beta)\right]^{-(n-r)} f_X(y_{(a)};\lambda,\beta), \quad (7)$$

where,

$$D_1(a) = a\binom{n-r}{a}. \quad (8)$$

Substituting (1) and (2) in (7) we obtain for the $Chen(\lambda,\beta)$ model

$$h_Y(y_{(a)};\lambda,\beta) = D_1(a)\sum_{\ell=0}^{a-1} \omega\_(a)\,(\ell)\;\; \beta\,\lambda\, y_{(a)}^{(\beta-1)} \exp\left\{y_{(a)}^{\beta} - \lambda\, b_a(\ell)(e^{y_{(a)}^{\beta}} - e^{x_{(r)}^{\beta}})\right\}, y_{(a)} > x_{(r)}, \quad (9)$$

where,

$$b_a(\ell) = n - r - a + \ell + 1 \quad (10)$$

and $\omega_a(\ell)$ is the same as $\omega_s(k)$, given in (3), with a and ℓ instead of s and k .

The Bayesian predictive density function of Y = y_ ((a)) is given by (see Dunsmore (1974))

$$h_1^*(y_{(a)}|\underline{x}) = \int h_Y(y_{(a)}|\lambda,\beta)\; g_1^*\,(\lambda|\beta,\underline{x})d\lambda, y_{(a)} > x_{(r)}, \quad (11)$$

where, $h_Y(y_{(a)}|\lambda,\beta)$ is the conditional pdf of $Y = y_{(a)} \equiv x_{(r+a)}, a = 1,2,\cdots,n-r$ and $g_1^*(\lambda|\beta,\underline{x})$ is the posterior density of the parameters λ and β .

Using (5), (9), and (11), we have

$$h_1^*(y_{(a)}|\underline{x}) = D_1(a)C_1\Gamma(d_0+1)\sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1}\frac{\omega_a(\ell)\omega_s(k)\beta y_{(a)}^{\beta-1}e^{y_{(a)}^{\beta}}}{\left[b_a(\ell)\left(e^{y_{(a)}^{\beta}}-e^{x_{(r)}^{\beta}}\right)+T(x)+b_1\right]^{d_0+1}}, \quad (12)$$

where, $\omega_s(k)$ and $T(\underline{x})$ are as given in (3), while $D_1(a)$ and $b_a(\ell)$ are as given, respectively, by (8) and (10).

Bayesian prediction bounds for the future order statistics $y_{(a)} \equiv x_{(r+a)}$ are obtained by evaluating $Pr[Y_{(a)} \geq \nu_1|\alpha;\, \underline{x}]$ for some ν_1. It follows from (12), that

$$Pr[Y_{(a)} \geq \nu_1|\alpha;\underline{x}] = \int_{\nu_1}^{\infty} h_1^*\;(\mathrm{y}_{(a)}|\underline{x})\mathrm{dy}_{(a)} = D_1(a)C_1\Gamma(d_0)\sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1}\frac{\omega_a(\ell)\,\omega_s(k)}{b_a(\ell)\left[b_a(\ell)\left(e^{\nu_1^{\beta}}-e^{x_{(r)}^{\beta}}\right)+T(\underline{x})+b_1\right]^{d_0}}. \quad (13)$$

A $100\,(1-\tau)\,\%$ Bayesian prediction interval for $y_{(a)} \equiv x_{(r+a)}$ is such $Pr[L(\underline{x}) \leq Y_{(a)} \leq U(\underline{x})] = 1 -$

τ, where $L(\underline{x})$ and $U(\underline{x})$ are the lower and upper bounds for $y_{(a)} \equiv x_{(r+a)}$. Thus equating (13) to $1-\frac{\tau}{2}$) and $\frac{\tau}{2}$, we obtain

$$D_1(a)C_1\Gamma(d_0)\sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1}\frac{\omega_a(\ell)\omega_s(k))}{b_a(\ell)\left[b_a(\ell)\left(e^{L(\underline{x})^{\beta}}-e^{x_{(r)}^{\beta}}\right)+T(\underline{x})+b_1\right]^{d_0}}=1-\frac{\tau}{2} \quad (14)$$

and

$$D_1(a)C_1\Gamma(d_0)\sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1}\frac{\omega_a(\ell)\omega_s(k))}{b_a(\ell)\left[b_a(\ell)\left(e^{U(\underline{x})^{\beta}}-e^{x_{(r)}^{\beta}}\right)+T(\underline{x})+b_1\right]^{d_0}}=\frac{\tau}{2}. \quad (15)$$

2.2. Bayesian Two-Sample Prediction

Suppose that we have two samples, the informative one (past sample) is a doubly Type-II censored sample, while the second one (future sample) is an ordered sample from the same population. Based on the past sample, we would like to obtain Bayesian prediction bounds for the b^{th} observation $Z_{(b)}, b=1,2,\cdots,m$. To do this, let Z_b be the b^{th} ordered observation in the future sample of size m whose pdf is given by (\ref (e1)). The conditional density function of $Z_{(b)}$ for a given parameter λ and β is

$$h_Z\left(z_{(b)};\lambda,\beta\right)=D_2(b)\left[1-F_X\left(z_{(b)};\lambda,\beta\right)\right]^{m-b}\left[F_X\left(z_{(r)};\lambda,\beta\right)\right]^{b-1}f_X\left(z_{(b)};\lambda,\beta\right), \quad (16)$$

where,

$$D_2(b)=b\binom{m}{b}. \quad (17)$$

Applying (1) and (2), in (16), we get

$$h_Z\left(z_{(b)};\lambda,\beta\right)=D_2(b)\sum_{j=0}^{b-1}\omega_b(j)\lambda\,\beta\; z_{(b)}^{\beta-1}exp\left\{z_{(b)}^{\beta}-\lambda\, b_b^*(j)\,(e^{z_{(b)}^{\beta}}-1)\right\}, z_{(b)}>0, \quad (18)$$

where,

$$b_b^*(j)=m-b+j+1 \quad (19)$$

and

$\omega_b(j)$ is given in (3) with b and j replacing s and ℓ.

Applying the posterior density function (18) and the conditional probability density function (18) in (11) the Bayesian predictive density function of $z_{(b)}, b = 1,2,\cdots,m$ is then given by

$$h_2^*\left(z_{(b)}|\underline{x}\right)=\int_0^{\infty}h_Z\;\left(z_{(b)};\lambda,\beta\right)g_1^*\left(\lambda|\beta,\underline{x}\right)d\beta=D_2(b)C_1\Gamma(d_0+1)\sum_{j=0}^{b-1}\sum_{k=0}^{s-1}\frac{\omega_s(k)\omega_b(j)\lambda\,\beta\,e^{z_{(b)}^{\beta}}}{\left[b_b^*(j)\left(e^{z_{(b)}^{\beta}}-1\right)+T(\underline{x})+b_1\right]^{d_0+1}}, \quad (20)$$

where, d_0 is given in (6) and $b_b^*(j)$ is given by (19).

Bayesian prediction bounds for the future order statistics $z_{(b)}, b = 1,2,\cdots,m$, are obtained by evaluating $Pr\left[Z_{(b)}\geq \nu_2|\lambda;\ \underline{x}\right]$ for some ν_2. It follows from (\ref (e28)) that

$$Pr\left[Z_{(b)} \geq \nu_2 \middle| \lambda;\ \underline{x}\ \right] = \int_{\nu_2}^{\infty} h_2^*\left(z_{(b)}\middle|\underline{x}\right) dz_{(b)}\ = D_2(b) C_1 \Gamma(d_0) \sum_{j=0}^{b-1} \sum_{k=0}^{s-1} \frac{\omega_s(k)\omega_b(j)}{b_b^*(j)\left[b_b^*(j)\left(e^{z_{(b)}^{\beta}} - 1\right) + T(\underline{x}) + b_1\right]^{d_0}}. \quad \textbf{(21)}$$

A $100\ (1-\tau)\%$ Bayesian prediction interval for $z_{(b)}, b\ =\ 1, 2, \cdots, m$ is such $\Pr\left(L(\underline{x}) \leq\ Z_{(b)} \leq\ U(\underline{x})\right) =\ 1 - \tau$, where $L(\underline{x})$ and $U(\underline{x})$ are the lower and upper bounds for $z_{(b)}, b\ =\ 1, 2, \cdots, m$. Thus equating (\ref(e29)) to $1\ -\frac{\tau}{2}$ and $\frac{\tau}{2}$, we obtain

$$D_2(b) C_1 \Gamma(d_0) \sum_{\ell=0}^{a-1} \sum_{k=0}^{s-1} \frac{\omega_s(k)\omega_b(\ell)}{b_b^*(j)\left[b_b^*(j)\left(e^{L(\underline{x})^{\beta}} - 1\right) + T(\underline{x}) + b_1\right]^{d_0}} = 1 - \frac{\tau}{2} \quad \textbf{(22)}$$

and

$$D_2(b) C_1 \Gamma(d_0) \sum_{\ell=0}^{a-1} \sum_{k=0}^{s-1} \frac{\omega_s(k)\omega_b(\ell)}{b_b^*(j)\left[b_b^*(j)\left(e^{U(\underline{x})^{\beta}} - 1\right) + T(\underline{x}) + b_1\right]^{d_0}} = \frac{\tau}{2}. \quad \textbf{(23)}$$

3. TWO-PARAMETER CASE

This section is concerned with Bayesian prediction of future observations of the Chen distribution when both of the two parameters λ and β are unknown. This bivariate prior density function was used by Sarhan et al. (2012) in their parameter estimation for a tow-parameter bathtub-shaped lifetime distribution.

Suppose that the prior belief of the experimenter is measured by the bivariate prior density function for λ and, given by

$$g(\lambda, \beta) =\ g_1(\lambda) g_2(\beta), \quad \textbf{(24)}$$

where, $g_1(\lambda)$ is a conjugate prior given by (4) and

$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\ \beta^{a_2-1}\ e^{-b_2\,\beta}, \beta\ >\ 0, (a_2, b_2\ >\ 0). \quad \textbf{(25)}$$

Using the likelihood function given by (3) and the joint prior density function given by (24), the joint posterior density function of \lambda and \beta is

$$g_2^*\ (\lambda, \beta, |\underline{x}) = C_2\ \lambda^{r-s+a_1} \beta^{r-s+a_2} \sum_{k=0}^{s-1} \omega_s(k)\ \prod_{i=s}^{r} x_i^{\beta-1}\ \exp\left\{\sum_{i=s}^{r} x_{(i)}^{\beta}\ - b\_2\ \beta - \lambda\left[T(\underline{x}) + b_1\right]\right\}, \quad \textbf{(26)}$$

where, C_2 is the normalizing constant given by

$$C_2^{-1} = \Gamma(d_0) \sum_{k=0}^{s-1} \omega_s(k)\ \ I_0(\underline{x}), \quad \textbf{(27)}$$

where, d_0 is given in (6) and

$$I_0(\underline{x}) = \int_0^{\infty} \frac{\beta^{r-s+a_2}}{[T(\underline{x}) + b_1]^{d_0}} \exp\left\{(\beta - 1) \sum_{i=s}^{r} \ln x_{(i)} + \sum_{i=s}^{r} x_{(i)}^{\beta} - b_2 \beta\ \right\} d\beta. \quad \textbf{(28)}$$

3.1. Bayesian One-Sample Predictions

Similarly and as before, let further, $y_{(a)} \equiv\ x_{(r+a)}, 1 \leq\ a \leq\ n - r$ be the a^{th} unobserved lifetime, which needs to be predicted. The conditional density of $y_{(a)}$ is given by (9). Applying the conditional density function given by (9) and the posterior density function given by (26), the Bayes predictive density function of $y_{(a)}$ is then given by

$$h_3^*(y_{(a)}|\underline{x}) = \int_0^\infty \int_0^\infty h_Y(y_{(a)}|\lambda,\beta)\ g_2^*(\lambda,\beta|\ \underline{x})\ d\lambda\ d\beta =$$

$$\Gamma(d_0+1)\ C_2\ D_1(a) \sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1} \omega_a(\ell)\ \omega_s(k) I_1(y_{(a)},\underline{x}), \quad \textbf{(29)}$$

where,

$$I_1(y_{(a)},\underline{x}) = \int_0^\infty \frac{\beta^{r-s+a_2+1}\ y_{(a)}^{\beta-1} \exp\{(\beta-1)\sum_{i=s}^{r} \ln\ x_{(i)} + \sum_{i=s}^{r} x_{(i)}^{r} + y_{(a)}^{\beta} - b_2\beta\}}{\left[b_a(\ell)\left(e^{y_{(a)}^{\beta}} - e^{x_{(r)}^{\beta}}\right) + b_1 + T(\underline{x})\right]^{d_0+1} d\beta} d\beta$$

Bayesian prediction bounds of $Y_{(a)}$ are obtained by evaluating $Pr[Y_{(a)} \geq \nu_3 |\ \underline{x}]$ for some ν_3 . It follows from (29)

$$\Pr[Y_{(a)}\ \geq \nu_3|\underline{x}] = \int_{\nu_3}^{\infty} h_3^*(y_{(a)}|\underline{x}) dy_{(a)}\ = \Gamma(d_0) C_2\ D_1(a) \sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1} \frac{\omega_a(\ell)\omega_s(k)}{b_a(\ell)}\ I_2(\nu_3,\underline{x}), \quad \textbf{(30)}$$

where,

$$I_2(\nu_3,\underline{x}) = \int_0^\infty \frac{\beta^{r-s+a_2} \exp\{(\beta-1)\sum_{i=s}^{r} \ln x_{(i)} + \sum_{i=s}^{r} x_{(i)}^{\beta} - b_2\beta\}}{\left[b_a(\ell)\left(e^{\nu_3^{\beta}} - e^{x_{(r)}^{\beta}}\right) + b_1 + T(\underline{x})\right]^{d_0}} d\beta. \quad \textbf{(31)}$$

A 100 $(1-\tau)$ % Bayesian prediction interval for $y_{(a)} \equiv x_{(r+a)}$ is such $Pr[L(\underline{x}) \leq Y_{(a)} \leq U(\underline{x})] = 1-\tau$, where $L(\underline{x})$ and $U(\underline{x})$ are the lower and upper bounds for $y_{(a)} \equiv x_{(r+a)}$. Thus equating (30) to $1-\frac{\tau}{2}$ and $\frac{\tau}{2}$, we obtain

$$\Gamma(d_0) C_2\ D_1(a) \sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1} \frac{\omega_a(\ell)\omega_s(k)}{b_a(\ell)} I_2(L(\underline{x}),\underline{x}) = 1-\frac{\tau}{2} \quad \textbf{(32)}$$

and

$$\Gamma(d_0) C_2\ D_1(a) \sum_{\ell=0}^{a-1}\sum_{k=0}^{s-1} \frac{\omega_a(\ell)\omega_s(k)}{b_a(\ell)} I_2(U(\underline{x}),\underline{x}) = \frac{\tau}{2}, \quad \textbf{(33)}$$

where, $I_2(L(\underline{x}),\underline{x})$ and $I_2(U(\underline{x}),\underline{x})$ are given by (31) with ν_3 being replaced, respectively, by $L(\underline{x})$ and $U(\underline{x})$.

3.2. Bayesian Two-Sample Prediction

As before, assume that Z_b is the b^{th} ordered lifetime in a future, unobserved, sample of m components whose lifetimes follow the $Chen(\lambda,\beta)$ distribution given by (1). Thus, the density function of Z_b is given by (18). Applying the conditional and posterior density functions given by (18) and (26), respectively, the Bayesian predictive density function of Z_b , $b\ = 1,2,\dots,m$, is given by

$$h_4^*(Z_{(b)}|\underline{x})\ = \int_0^\infty \int_0^\infty h_Z(z_{(b)}|\lambda,\beta)\ g_2^*(\lambda,\beta|\ \underline{x})\ d\lambda\ d\beta$$

$$= \Gamma(d_0+1) C_2\ \mathrm{D}_2\ (\mathrm{b}) \sum_{\mathrm{j}=0}^{\mathrm{b}-1} \omega_b(j)\omega_s(k) I_3(z_{(b)},\underline{x}), \quad \textbf{(34)}$$

where,

$$I_3(z_{(b)},\underline{x}) = \int_0^\infty \frac{\beta^{r-s+a_2} z_{(b)}^{\beta-1} \exp\{(\beta-1)\sum_{\mathrm{i=s}}^{\mathrm{r}} \ln \mathrm{x_{(i)}} + \sum_{\mathrm{i=s}}^{\mathrm{r}} \mathrm{x}_{(\mathrm{i})}^{\beta} - b_2\beta\}}{\left[b_b^*(j)\left(e^{z_{(b)}^{\beta}} - 1\right) + b_1 + T(\underline{x})\right]^{d_0}} d\beta$$

Bayesian prediction bounds for the future order statistics $z_{(b)}$, where $b\ =\ 1,2,\cdots,m$, are obtained by evaluating $Pr[z_{(b)} \geq \nu_4 |\ \underline{x}\ ]$ for some ν_4 . It follows from (34) that

$\Pr[Z_{(b)} \geq \nu_4 | \underline{x}] = \int_{\nu_4}^{\infty} h_4^*(z_{(b)}|\underline{x})\ dz_{(b)}$

$$= \Gamma(d_0) C_2\, D_2\, (b) \sum_{j=0}^{b-1} \sum_{k=0}^{s-1} \frac{\omega_b(j)\omega_s(k)}{b_b^*(j)} I_4(\nu_4, \underline{x}), \quad \textbf{(35)}$$

where

$$I_4(\nu_4, \underline{x}) = \int_0^{\infty} \frac{\beta^{r-s+a_2-1} \exp\left\{(\beta-1)\sum_{i=s}^{r} \ln x_{(i)} + \sum_{i=s}^{r} x_{(i)}^{\beta} - b_2\beta\right\}}{\left[b_b^*(j)\left(e^{\nu_4^{\beta}}-1\right)+b_1+T(\underline{x})\right]^{d_0}} d\beta. \quad \textbf{(36)}$$

A $100\,(1-\tau)\,\%$ Bayesian prediction interval for $z_{(a)}$ is such $Pr[L(\underline{x}) \leq Z_{(b)} \leq U(\underline{x})] = 1-\tau$, where, $L(\underline{x})$ and $U(\underline{x})$ are the lower and upper bounds for$z_{(b)}$.

Thus equating (35) to $1 - \frac{\tau}{2}$ and $\frac{\tau}{2}$, we obtain

$$\Gamma(d_0) C_2\, D_2\, (b) \sum_{j=0}^{b-1} \sum_{k=0}^{s-1} \frac{\omega_b(j)\omega_s(k)}{b_b^*(j)} I_4(L(\underline{x}), \underline{x}) = 1 - \frac{\tau}{2} \quad \textbf{(37)}$$

and

$$\Gamma(d_0) C_2\, D_2\, (b) \sum_{j=0}^{b-1} \sum_{k=0}^{s-1} \frac{\omega_b(j)\omega_s(k)}{b_b^*(j)} I_4(U(\underline{x}), \underline{x}) = 1 \frac{\tau}{2} \quad \textbf{(38)}$$

where, $I_4(L(\underline{x}), \underline{x})$ and $I_4(U(\underline{x}), \underline{x})$ are given by (36) with ν_4 being replaced, by $L(\underline{x})$ and$U(\underline{x})$, respectively.

4. NUMERICAL ILLUSTRATIONS

This section is devoted to illustrating both the one- and two-sample prediction assuming that the (unknown) two-parameter case. Where, two numerical examples are given to illustrate the results of Sections 3.1 and 3.2. The results related to one-parameter case may be similarly dealt with.

4.1. Example 1: (One-Sample Prediction)

The 95 % Bayesian prediction bounds for the remaining $(n-r)$ order statistics $s = 1, 2, \cdots, n-r$, are obtained according to the following steps:

1. For given values of the hyper parameters a_1 and b_1 a generated value of λ is obtained from the prior distribution with pdf (4).
2. For a given value of the hyper parameters a_2 and b_2 a generated value of β is obtained from the prior distribution with pdf (25).
3. Using the generated values of \lambda and β from Steps 1 and 2, a sample of size n is generated from the $Chen(\lambda, \beta)$ distribution with pdf, which is given by (1).
4. Using some sorting routine, a doubly Type-II censored sample of size $(r-s+1)$ from the Chen (\lambda, \beta) distribution is then obtained.
5. Based on the above generated doubly Type-II censored samples of size $(r-s+1)$, the 95 % Bayesian prediction bounds for the remaining $(n-r)$ ordered values$x_{(r+1)}, x_{(r+2)}, \cdots, x_{(n)}$, are then numerically calculated by solving Equations (32) and (33).

In this example, three different values for the sample size, namely, $n = 20, 30\ or\ 40$, and the hyperparamters a_1, b_1, a_2 and b_2 are chosen to be $2, 3, 2$ and 2, respectively. This particular choice of the hyperparamters suggests that $\lambda = 1.1552$ and $\beta = 1.7327$. Table 1 presents 95 % Bayesian prediction intervals for $x_{(r+s)} \equiv y_s, s = 1, \cdots, 5$, and their corresponding lengths.

The percentage coverage of each $y_s, s = 1, \cdots, 5$, can be obtained by generating 10,000 future samples each of size $n - r = 5$, from the same $Chen(\lambda, \beta)$ distribution with cdf given by (1), such that $y_1 > x_r$, then calculate the actual predicted levels of y_s , where $s = 1, 2, \cdots, 5$. These percentage coverages are also displayed in Table 1.

Table 1: 95 % Bayesian Prediction Intervals for $y_1, y_2, \cdots, y_5$

r		y_1	y_{12}	y_3	y_4	y_5
15	1*	95.20 %	95.85%	96.63%	97.66%	98.27%
	2*	(0.4196, 0.6592)	(0.4358,0.8052)	(0.4678,0.9665)	(0.5167,1.1767)	(0.5948,1.5357)
	3*	0.2396	0.3694	0.4986	0.6600	0.9409
25	1*	96.38%	96.94%	97.24%	97.53%	97.76%
	2*	(0.7861,0.9827)	(0.7982,1.1120)	(0.8222,1.2583)	(0.8594,1.4515)	(0.9203,1.7839)
	3*	0.1967	0.3139	0.4361	0.5921	0.8635
35	1*	95.64%	96.26%	97.04%	97.74%	97.69%
	2*	(0.8021,0.9724)	(0.8130,1.0830)	(0.8349,1.2078)	(0.8692,1.3729)	(0.9261,1.6594)
	3*	0.1703	0.2700	0.3729	0.5037	0.7333

1* Simulated prediction levels of $y_s, s = 1, \cdots, 5$.

2* Bayesian prediction intervals for $y_s, s = 1, \cdots, 5$.

3* Length of the Bayesian prediction intervals

4.2. Example 2: (Two-Sample Prediction)

In this example, an "observed" doubly Type-II censored sample, $x_{(s+1)}, x_{(s+2)}, \cdots, x_{(r)}$, is generated by using Steps 1-4, as given in Section (4.1). The values of n, a_1, b_1, a_2 and b_2 are chosen as in the above example. Based on these values and the generated sample, the 95 % Bayesian prediction bounds for a future "unobserved" sample of size $m = 5$, $z_{(1)}, z_{(2)}, \cdots, z_{(5)}$, are obtained by solving Equations (37) and (38), separately. The 95% Bayesian prediction intervals for $z_{(k)}, k = 1, \cdots, 5$, are presented in Table (2) together with their corresponding lengths.

The percentage coverage of each $z_k, k = 1, \cdots, 5$, can be obtained by generating 10,000 future samples each of size m=5 , from the same $Chen(\lambda, \beta)$ distribution with cdf given by (1), then calculate the actual prediction levels of z_k , where $k = 1, 2, \cdots, 5$. These percentage coverages are also displayed in Table 2.

Table 1: 95 % Bayesian Prediction Intervals for $z_1, z_2, \cdots, z_5$

r		z_1	z_2	z_3	z_4	z_5
15	1*	95.97%	96.09%	96.08%	96.06%	97.17%
	2*	(0.0056, 0.4291)	(0.0432,0.6038)	(0.1089, 0.7816)	(0.2020, .0059)	(0.3388, .3876)
	3*	0.4235	0.5606	0.6727	0.8039	1.0488
25	1*	96.53%	96.46%	95.98%	95.70%	95.56%
	2*	(0.0071, 0.4978)	(0.0549, 0.6961)	(0.1372, 0.8926)	(0.2517,1.1338)	(0.4166,1.5315)
	3*	0.4907	0.6412	0.7554	0.8821	1.1149
35	1*	96.04 %	96.00%	95.92%	95.49%	95.86%
	2*	(0.0073, 0.4602)	(0.0540, 0.6481)	(0.1318, 0.8332)	(0.2389,0.0571)	(0.3936,1.4197)
	3*	0.4529	0.5941	0.7014	0.8182	1.0261

1* Simulated prediction levels of $z_k, k = 1, \cdots, 5$.

2* Bayesian prediction intervals for $z_k, k = 1, \cdots, 5$.

3* Length of the Bayesian prediction intervals

5. CONCLUSIONS

In this article, Bayesian prediction bounds are obtained for future observations from the two parameter $Chen(\lambda, \beta)$ distribution. It has been noticed from Tables 1 and 2 that, which prediction intervals are affected by increasing n, and in this case, the coverage probabilities are quite close to the confidence levels 95 %, and therefore the intervals tend to perform very well in terms of simulated coverage probabilities. The Bayesian prediction intervals for the smallest and the largest future ordered lifetimes, which are practically of some special interest, are considered in the simulation. The aim of this simulation is to show how good is the given Bayesian prediction intervals for the future lifetimes. The simulated percentage converges are all quite close to the nominated ones.

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REFERENCES

1. Ahmad, A. A. (2011). On Bayesian interval prediction of future generalized order statistics using doubly censoring. Statistics 45, 413 - 425.
2. Aitchison, J. and Dunsmore, I. R. (1975). Statistical Prediction Analysis. Cambridge University Press Cambridge, UK.
3. AL-Hussaini, E. K. Abdel-Hamid, A. H. and Hashem, A. F. (2015). Bayesian prediction intervals of order statistics based on progressively type-II censored competing risks data from the half-logistic distribution. Journal of the Egyptian Mathematical Society 23, 190 - 196.
4. AL-Hussaini, E. K. and Ahmad, A. A.(2003). On Bayesian interval prediction of future records. Test 12, 79 - 99.
5. Ateya, S. F. (2013). Bayesian Prediction Intervals of Future Generalized Order Statistics under a Finite Mixture of Generalized Exponential Distributions. Applied Mathematical Sciences 7 (32), 1575-1592.
6. Arnold, B. C. Balakrishnan, N. and Nagaraja, H. N. (1992). A First Course in Order Statistics. John Wiley and Sons, New York.
7. Balakrishnan, N. and Shafay, A. R. (2012). One- and two-sample Bayesian prediction
8. Intervals based on Type-II hybrid censored data. Communications in Statistics Theory and Methods Communications in Statistics Theory and Methods 41 (9), 1511- 1531.
9. Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Statistics and Probability Letters 49, 155-162.

10. Dunsmore, I. R. (1974). The skew-normal distribution and related multivariate families. Technometrics 16, 455 - 460.

11. Geisser, S. (1993). Predictive Inference: An Introduction. Chapman and Hall, London

12. Howlader, H. A. Hossain, A. M. and Makhnin, O. (2007). Bayesian Prediction and Estimation from Pareto Distribution of the First Kind. International Journal of Statistical Sciences 6, 137-150.

13. Jaheen, Z. F. (2002). On Bayesian prediction of generalized order statistics. J. Statist. Theor. and Appli.,1 (3), 191- 204.

14. Mohie El-Din, M. M. and Shafay, A. R. (2013). One- and two-sample Bayesian prediction intervals based on progressively Type-II censored data, Statistical Papers 54, 287--307.

15. Niazi, S. F. (2016). Prediction intervals based on Gompertz doubly censored data. Computational Statistics. 31 227-246.

16. Niazi, S. F.and Abd-Elrahman, A. M. (2015). Bayesian prediction bounds for a two-parameter lifetime distribution with bathtub-shaped failure rate function. SYLWAN 159 (6), 34-50.

17. Sarhan, M. A. Hamilton, D. C. and Simeth, B. (2011). Parameter estimation for a two-parameter bathtub-shaped lifetime distribution. Applied Mathematical Modeling 36 (11), 5380--5392.